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# Existence and Uniqueness of Solutions to First-Order Multipoint Boundary Value Problems

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**Abstract**—In this paper, we give existence and uniqueness results for solutions of multipoint boundary value problems of the form

$$x' = f(t, x(t)) + e(t), \qquad t \in (0,1),$$
$$\sum_{j=1}^{m} A_j x(\eta_j) = 0,$$

where $f : [0,1] \times R^n \to R^n$ is a Carathéodory function, $A_j$s ($j = 1, 2, \dots, m$) are constant square matrices of order $n$, $0 \le \eta_1 < \eta_2 < \dots < \eta_{m-1} < \eta_m \le 1$, and $e(t) \in L^1([0,1], R^n)$. The existence of solutions is proven by the coincidence degree theory. As an application, we also give one example to demonstrate our results. 

**Keywords**—First-order system, Existence and uniqueness of solution, Multipoint boundary value problems, Fredholm operator, Coincidence degree.

## 1. INTRODUCTION

We are interested in the existence of solutions of the following multipoint boundary value problem (BVP):

$$x' = f(t, x(t)) + e(t), \qquad t \in (0,1), \tag{1.1}$$

$$\sum_{j=1}^{m} A_j x(\eta_j) = 0, \tag{1.2}$$

where $f : [0,1] \times R^n \to R^n$ is a Carathéodory function, $A_j$s ($j = 1, 2, \dots, m$) are constant square matrices of order $n$, $0 \le \eta_1 < \eta_2 < \dots < \eta_{m-1} < \eta_m \le 1$, and $e(t) \in L^1([0,1], R^n)$ are given.

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When $m = 3$ and $e(t) \equiv 0$, the above BVP was very recently studied by Ma [1] using the Leray-Schauder continuation theorem. However, the existence results in [1] mainly depend upon a restrictive condition, i.e.,

$$\det(A_1 + A_2 + A_3) \neq 0.$$

It is therefore natural to ask whether similar existence results can be obtained if $\det(A_1 + A_2 + A_3) = 0$. So in this paper, we will derive existence results for BVP (1.1),(1.2) when $\sum_{j=1}^{m} A_j = 0$, which gives a partial answer to the questions stated above. The key tool in our approach is based upon the coincidence degree theory of Mawhin [2,3].

We remark that there are a number of studies concerned with the existence of solutions to second-order multipoint boundary value problems; see, for example, [4–8] and references therein. However, to our knowledge, few papers can be found dealing with multipoint boundary value problems of first-order systems except for the works due to Ma [1] and Murty and Sivasundaram [9]. In [9], Murty and Sivasundaram studied the existence and uniqueness of solutions to BVP (1.1),(1.2) (when $m = 3$, $e(t) \equiv 0$) using the Banach contraction mapping principle, and the key conditions in [9] depend on the fundamental matrix of the variational system of (1.1). This requires $f$ to be a continuously differentiable function.

Now, we will briefly recall some notation and an abstract existences result.

Let $Y, Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by $K_p$. If $\Omega$ is an open bounded subset of $Y$ such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \to Y$ is compact.

The theorem we use is Theorem 2.4 of [2] or Theorem IV.13 of [3].

THEOREM A. *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial\Omega)] \times (0, 1)$;*
(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*
(iii) *$\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q : Z \to Z$ is a projection as above with $\operatorname{Im} L = \operatorname{Ker} Q$.*

*Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.*

Throughout this paper, the function $g : [0, 1] \times R^n \to R^n$ is a Carathéodory function, which means

(i) $g(t, \cdot)$ is continuous on $R^n$ for a.e. $t \in [0, 1]$,
(ii) $g(\cdot, x)$ is Lebesgue measurable on $[0, 1]$, for each $x \in R^n$,
(iii) for each $\rho > 0$, there exists an $h_\rho \in L^1([0, 1], R^n)$ such that

$$|f_i(t, x)| \leq (h_\rho)_i(t), \qquad \text{for a.e. } t \in [0, 1], \quad \|x\| < \rho, \quad \text{and} \quad i = 1, \dots, n.$$

As usual, the notation we will use herein is mostly standard. We denote the $n \times n$ identity matrix by $E$, the Banach space of all constant square matrices of order $n$ by $M_{n \times n}$ with the norm

$$\|B\| = \max_{1 \leq i,j \leq n} |b_{i,j}|.$$

For $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in R^n$, define

$$\|\alpha\| = \max_{1 \leq i \leq n} |\alpha_i|.$$

The corresponding $L^1$-norm in $L^1([0,1],R^n)$ is defined by

$$\|x\|_1 = \max_{1\le i\le n} \int_0^1 |x_i(t)|\,dt.$$

The $L^\infty$-norm in $C([0,1],R^n)$ is

$$\|x\|_\infty = \max_{1\le i\le n} \sup_{t\in[0,1]} |x_i(t)|.$$

Furthermore, in this paper, we always assume the following:

(H) $\sum_{j=1}^m A_j = 0$ and $\det(\sum_{j=1}^m A_j\eta_j) \neq 0$.

## 2. MAIN RESULTS

In this section, we shall prove the existence and uniqueness results for solutions of BVP (1.1),(1.2).

Let $Y = C([0,1],R^n)$, $Z = L^1([0,1],R^n)$. Define $L$ to be the linear operator from $\operatorname{dom} L \subset Y$ to $Z$ with

$$\operatorname{dom} L = \left\{ x \in Y : \sum_{j=1}^m A_j x(\eta_j) = 0,\ x' \in L^1([0,1],R^n) \right\},$$

and $Lx = x'$, $x \in \operatorname{dom} L$. We define $N : Y \to Z$ by setting

$$Nx = f(t,x(t)) + e(t), \qquad t \in [0,1],$$

then BVP (1.1),(1.2) can be written

$$Lx = Nx.$$

LEMMA 2.1. *Let (H) hold, then $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projection operator $Q : Z \to Z$ can be defined by*

$$Qy = \left(\sum_{j=1}^m A_j\eta_j\right)^{-1} \sum_{j=1}^m A_j \int_0^{\eta_j} y(s)\,ds,$$

*and the linear operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by*

$$K_p y = \int_0^t y(s)\,ds.$$

*Also,*

$$\|K_p y\|_\infty \le \|y\|_1, \qquad \text{for all } y \in \operatorname{Im} L.$$

PROOF. It is clear that $\operatorname{Ker} L = \{x \in \operatorname{dom} L : x = d,\ d \in R^n\}$. We now show that

$$\operatorname{Im} L = \left\{ y \in Z : \sum_{j=1}^m A_j \int_0^{\eta_j} y(s)\,ds = 0 \right\}. \tag{2.1}$$

Since the problem

$$x' = y \tag{2.2}$$

has a solution $x(t)$, that satisfies $\sum_{j=1}^m A_j x(\eta_j) = 0$ if and only if

$$\sum_{j=1}^m A_j \int_0^{\eta_j} y(s)\,ds = 0. \tag{2.3}$$

In fact, if (2.2) has a solution $x(t)$ such that $\sum_{j=1}^{m} A_j x(\eta_j) = 0$, then from (2.2), we have

$$x(t) = x(0) + \int_0^t y(s)\,ds.$$

In view of condition (H), we obtain

$$0 = \sum_{j=1}^{m} A_j x(\eta_j) = \sum_{j=1}^{m} A_j \int_0^{\eta_j} y(s)\,ds,$$

that is,

$$\sum_{j=1}^{m} A_j \int_0^{\eta_j} y(s)\,ds = 0.$$

On the other hand, since (2.3) holds, we can set

$$x(t) = d + \int_0^t y(s)\,ds,$$

where $d \in R^n$ is arbitrary. Hence, $x(t)$ is a solution of (2.2) and $\sum_{j=1}^{m} A_j x(\eta_j) = 0$. Therefore, (2.1) holds.

For $y \in Z$, consider the projection

$$Qy = \left(\sum_{j=1}^{m} A_j \eta_j\right)^{-1} \sum_{j=1}^{m} A_j \int_0^{\eta_j} y(s)\,ds.$$

Let $y_1 = y - Qy$, then $y_1 \in \operatorname{Im} L$ (since $\sum_{j=1}^{m} A_j \int_0^{\eta_j} y_1(s)\,ds = 0$). Hence, $Z = \operatorname{Im} L + R^n$, since $\operatorname{Im} L \cap R^n = \{0\}$. Thus,

$$\dim \operatorname{Ker} L = \dim R^n = \operatorname{co}\dim \operatorname{Im} L = n.$$

Hence, $L$ is a Fredholm operator of index zero.

Let $P : Y \to Y$ be defined by

$$Px = x(0).$$

Then the generalized inverse operator $Kp : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of $L$ can be written by

$$K_p y = \int_0^t y(s)\,ds.$$

In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = [(K_p y)(t)]' = y(t),$$

and for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we know

$$(K_p L)x(t) = K_p x'(t) = \int_0^t x'(s)\,ds = x(t) - x(0).$$

Since $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $x(0) = 0$, we have

$$(K_p L)x(t) = x(t).$$

This shows $K_p = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. Furthermore, it is clear that

$$\|K_p y\|_\infty \le \|y\|_1, \qquad \text{for all } y \in \operatorname{Im} L.$$

This completes the proof of Lemma 2.1.

THEOREM 2.1. *Let (H) hold. Let $f : [0,1] \times R^n \to R^n$ be a Carathéodory function, and assume the following.*

(H$_1$) *There exist functions $a, b, r \in L^1([0,1], R)$, and constant $\theta \in [0,1)$ such that for all $x \in R^n$, $t \in [0,1]$,*

$$\|f(t,x)\| \leq a(t)\|x\| + b(t)\|x\|^\theta + r(t). \tag{2.4}$$

(H$_2$) *There exists a constant $M > 0$ such that, for $x \in \operatorname{dom} L$, if there exist some $i_0 \in \{1,2,\dots,n\}$ such that $|x_{i_0}(t)| > M$ for all $t \in [0,1]$, then*

$$\sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s,x(s)) + e(s)]\,ds \neq 0. \tag{2.5}$$

(H$_3$) *There exists a constant $M^* > 0$ such that for any $d = (d_1,\dots,d_n)^\top \in R^n$, if $\|d\| > M^*$, then either*

$$d^\top \cdot \left(\sum_{j=1}^{m} A_j \eta_j\right)^{-1} \sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s,d) + e(s)]\,ds < 0 \tag{2.6}$$

*or*

$$d^\top \cdot \left(\sum_{j=1}^{m} A_j \eta_j\right)^{-1} \sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s,d) + e(s)]\,ds > 0. \tag{2.7}$$

*Then, for every $e \in L^1([0,1], R^n)$, BVP (1.1),(1.2) has at least one solution $x \in C([0,1], R^n)$ provided*

$$\|a\|_1 < \frac{1}{2}.$$

PROOF. Set

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}.$$

Then, for $x \in \Omega_1$, $Lx = \lambda Nx$, so $\lambda \neq 0$, and $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$. Therefore,

$$\sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s,x(s))]\,ds = 0.$$

By (H$_2$), there exists $t_i \in [0,1]$ such that $|x_i(t_i)| \leq M$ for all $i \in \{1,2,\dots,n\}$. Since $x_i(0) = x_i(t_i) - \int_0^{t_i} x_i'(t)\,dt$, this implies $|x_i(0)| \leq M + \|x_i'\|_1$, and thus

$$\|x(0)\| \leq M + \|x'\|_1. \tag{2.8}$$

Again,

$$\|x'\|_1 = \|Lx\|_1 \leq \|Nx\|_1, \tag{2.9}$$

and from (2.8),(2.9), we obtain

$$\|x(0)\| \leq M + \|Nx\|_1. \tag{2.10}$$

Also for $x \in \Omega_1$, $x \in \operatorname{dom} L \setminus \operatorname{Ker} L$, then $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $LPx = 0$. Applying Lemma 2.1, we have

$$\|(I-P)x\|_\infty = \|K_p L(I-P)x\|_\infty \leq \|L(I-P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1. \tag{2.11}$$

From (2.10),(2.11), we obtain

$$\|x\|_\infty \leq \|Px\|_\infty + \|(I-P)x\|_\infty = \|x(0)\| + \|(I-P)x\|_\infty \leq 2\|Nx\|_1 + M. \tag{2.12}$$

In view of (2.4) and (2.12), we have

$$\|x\|_\infty \le 2\left[\|a\|_1\|x\|_\infty + \|b\|_1\|x\|_\infty^\theta + \|r\|_1 + \|e\|_1\right] + M,$$

and thus,

$$\|x\|_\infty \le \frac{2\|b\|_1}{1-2\|a\|_1}\|x\|_\infty^\theta + \frac{2}{1-2\|a\|_1}\left[\|r\|_1 + \|e\|_1 + \frac{M}{2}\right].$$

Since $\theta \in [0,1)$, from above the inequality, there exists $M_1 > 0$ such that

$$\|x\|_\infty \le M_1.$$

Therefore, $\Omega_1$ is bounded.

Let

$$\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, $x \in \operatorname{Ker} L = \{x \in \operatorname{dom} L : x = d,\ d \in R^n\}$, and $QNx = 0$, thus,

$$\sum_{j=1}^m A_j \int_0^{\eta_j} [f(s,d) + e(s)]\,ds = 0,$$

and hence, $\|d\| \le M$. Otherwise, if $\|d\| > M$, from ($H_2$), we obtain

$$\sum_{j=1}^m A_j \int_0^{\eta_j} [f(s,d) + e(s)]\,ds \ne 0,$$

which is a contradiction. Therefore, $\Omega_2$ is bounded.

Next, according to condition ($H_3$), for any $d \in R^n$, if $\|d\| > M^*$, then either (2.6) or (2.7) holds.

If (2.6) holds, set

$$\Omega_3 = \{x \in \operatorname{Ker} L : -\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by $\wedge(d) = d$, $\forall\, d \in R^n$.

Since any $x = d_0 \in \Omega_3$, we see

$$\lambda d_0 = (1-\lambda)\cdot\left(\sum_{j=1}^m A_j\eta_j\right)^{-1} \sum_{j=1}^m A_j \int_0^{\eta_j} [f(s,d_0) + e(s)]\,ds.$$

If $\lambda = 1$, then $d_0 = 0$. Otherwise, if $\|d_0\| > M^*$, in view of (2.6), we have

$$d_0^{\top}(1-\lambda)\cdot\left(\sum_{j=1}^m A_j\eta_j\right)^{-1} \sum_{j=1}^m A_j \int_0^{\eta_j} [f(s,d_0) + e(s)]\,ds < 0,$$

which contradicts $\lambda d_0^{\top}\cdot d_0 \ge 0$. Therefore, $\Omega_3 \subset \{x \in \operatorname{Ker} L : \|x\|_\infty < M^*\}$ is bounded.

If (2.7) holds, then set

$$\Omega_3 = \{x \in \operatorname{Ker} L : \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

(here $\wedge$ is the same as the above definition). Similar to the above argument, we see that $\Omega_3$ is bounded too.

In the following, we shall prove that all conditions of Theorem A are satisfied. Let $\Omega$ be a bounded open subset of $Y$ such that $\bigcup_{i=1}^3 \bar{\Omega}_i \subset \Omega$. By using the Ascoli-Arzéla theorem, we can prove that $K_p(I-Q)N : \bar{\Omega} \to Y$ is compact, thus $N$ is $L$-compact on $\bar{\Omega}$. Then by the above argument we have:

(i) $Lx \ne \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial\Omega)] \times (0,1)$;

(ii) $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

At last we will prove that (iii) of Theorem A is satisfied. Let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According to the above argument, we know

$$H(x,\lambda) \neq 0, \qquad \text{for } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Thus, by the homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(\pm\wedge, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

By Theorem A, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$, so that BVP (1.1),(1.2) has a solution in $C([0,1], R^n)$. This completes the proof.

In the following, under stronger hypotheses than what we had before, we are able to prove uniqueness of solutions to BVP (1.1),(1.2).

THEOREM 2.2. *Suppose that conditions* ($\mathrm{H}_1$) *and* ($\mathrm{H}_2$) *in Theorem 2.1 are replaced by the following conditions, respectively.*

($\bar{\mathrm{H}}_1$) *There exists functions $a \in L^1([0,1], R)$ such that for all $x, y \in R^n$, $t \in [0,1]$,*

$$\|f(t,x) - f(t,y)\| \leq a(t)\|x - y\|.$$

($\bar{\mathrm{H}}_2$) *For $x \in \operatorname{dom} L$, if there exist some $i_0 \in \{1,2,\dots,n\}$ such that $|x_{i_0}(t)| > 0$ for all $t \in [0,1]$, then*

$$\sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s,x(s)) + e(s)]\, ds \neq 0.$$

*Then, for every $e \in L^1([0,1], R^n)$, BVP (1.1),(1.2) has a unique solution $x \in C([0,1], R^n)$ provided*

$$\|a\|_1 < \frac{1}{2}.$$

PROOF. The existence of a solution of BVP (1.1),(1.2) follows immediately from Theorem 2.1 by setting $b(t) \equiv 0, r(t) = \|f(t,0)\|$, $t \in [0,1]$.

Now suppose that $x_1, x_2 \in C([0,1], R^n)$ are two solutions of BVP (1.1),(1.2), and write $x = x_1 - x_2$. Then we get

$$x'(t) = f(t, x_1(t)) - f(t, x_2(t)), \tag{2.13}$$

$$\sum_{j=1}^{m} A_j x(\eta_j) = 0. \tag{2.14}$$

Let $Y, Z, Q, P, L$ be as in the proof of Theorem 2.1, and

$$Nx(t) = f(t, x_1(t)) - f(t, x_2(t)).$$

Now, assuming that $x \neq 0$, in view of $Lx = Nx$, we have $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$, and hence,

$$\sum_{j=1}^{m} A_j \int_0^{\eta_j} [f(s, x_1(s)) - f(t, x_2(s))]\, ds = 0.$$

From ($\bar{\mathrm{H}}_2$), there exists $t_i \in [0,1]$ such that $x_i(t_i) = 0$ for all $i \in \{1,2,\dots,n\}$. Furthermore, $x_i(0) = x_i(t_i) - \int_0^{t_i} x_i'(t)\, dt$ implies

$$\|x(0)\| \leq \|x'\|_1 \tag{2.15}$$

and

$$\|x'\|_1 = \|Lx\|_1 \le \|Nx\|_1. \tag{2.16}$$

Hence, from (2.15),(2.16), we obtain

$$\|x(0)\| \le \|Nx\|_1. \tag{2.17}$$

Also, because $x \in \operatorname{dom} L \setminus \operatorname{Ker} L$, we know $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, and $LPx = 0$. From Lemma 2.1, we have

$$\|(I-P)x\|_\infty = \|K_pL(I-P)x\|_\infty \le \|L(I-P)x\|_1 = \|Lx\|_1 \le \|Nx\|_1. \tag{2.18}$$

Then (2.17),(2.18) yield

$$\|x\|_\infty \le \|Px\|_\infty + \|(I-P)x\|_\infty = \|x(0)\| + \|(I-P)x\|_\infty \le 2\|Nx\|_1. \tag{2.19}$$

In view of $(\bar{\mathrm{H}}_1)$ and (2.19), we have

$$\|x\|_\infty \le 2\|a\|_1\|x\|_\infty.$$

By our assumption, the coefficient on the right is less than 1, which is a contradiction. Thus, $x(t) = 0$ for $t \in [0,1]$, so that $x_1 = x_2$.

This completes the proof of the theorem.

Finally, in order to illustrate our result, we consider one example.

EXAMPLE 2.1. Consider the boundary value problems

$$\begin{aligned} x_1' &= \frac{1}{6}x_1\left(1+\cos^2 x_2\right) + 3\sin(x_1)^{1/3} + \cos^2 t + 1,\\ x_2' &= \frac{1}{6}x_2\left(1+e^{-\sin^2 x_1}\right) + 3\sin(x_2)^{1/3} + \sin^2 t + 1, \end{aligned} \tag{2.20}$$

$$\begin{aligned} -\frac{3}{2}x_1(0) + x_1\left(\frac{1}{2}\right) + \frac{1}{2}x_1(1) &= 0,\\ -3x_1(0) - 2x_2(0) + 2x_1\left(\frac{1}{2}\right) + x_2\left(\frac{1}{2}\right) + x_1(1) + x_2(1) &= 0. \end{aligned} \tag{2.21}$$

Let $\eta_1 = 0$, $\eta_2 = 1/2$, $\eta_3 = 1$, $f_1(t,x) = (1/6)x_1(1+\cos^2 x_2) + 3\sin(x_1)^{1/3}$, $f_2(t,x) = (1/6)x_2(1+e^{-\sin^2 x_1}) + 3\sin(x_2)^{1/3}$, $e_1(t) = \cos^2 t + 1$, $e_2(t) = \sin^2 t + 1$, $f(t,x) = (f_1(t,x), f_2(t,x))^\top$, $e(t) = (e_1(t), e_2(t))^\top$, and (2.21) can be written

$$A_1 \cdot (x_1(\eta_1), x_2(\eta_1))^\top + A_2 \cdot (x_1(\eta_2), x_2(\eta_2))^\top + A_3 \cdot (x_1(\eta_3), x_2(\eta_3))^\top = 0.$$

Hence,

$$A_1 + A_2 + A_3 = 0 \qquad \text{and} \qquad \det(A_1\eta_1 + A_2\eta_2 + A_3\eta_3) = \frac{3}{2} \ne 0,$$
$$\|f(t,x)\| \le \frac{1}{3}\|x\| + 3\|x\|^{1/3}, \qquad \text{for all } t \in [0,1].$$

Taking $a = 1/3$, then $\|a\|_1 = 1/3 < 1/2$. Again,

$$\begin{aligned} \sum_{j=1}^{3} A_j \int_0^{\eta_j} [f(s,x(s)) + e(s)]\,ds &= \left(\int_0^1 [f_1(s,x(s)) + e_1(s)]\,ds, \int_0^1 \left[2f_1(s,x(s)) \right.\right.\\ &\qquad \left.\left. + \frac{3}{2}f_2(s,x(s)) + 2e_1(s) + \frac{3}{2}e_2(s)\right] ds\right)^\top \\ &= \left(F_1(x_1,x_2,e_1,e_2), F_2(x_1,x_2,e_1,e_2)\right)^\top, \end{aligned} \tag{2.22}$$

where

$$F_1(x_1,x_2,e_1,e_2)=\int_0^1[f_1(s,x(s))+e_1(s)]\,ds,$$
$$F_2(x_1,x_2,e_1,e_2)=\int_0^1\left[2f_1(s,x(s))+\frac{3}{2}f_2(s,x(s))+2e_1(s)+\frac{3}{2}e_2(s)\right]ds.$$

Now take $M=31$, for any $x\in C([0,1],R^2)\cap L^1([0,1],R^2)$, and assume $|x_1(t)|>M$ holds for any $t\in[0,1]$. Since $x_1$ is continuous, then either $x_1(t)>M$ or $x_1(t)<-M$ hold any $t\in[0,1]$.

If $x_1(t)>M$ holds for any $t\in[0,1]$, then

$$\begin{aligned}F_1(x_1,x_2,e_1,e_2)&=\int_0^1[f_1(s,x(s))+e_1(s)]\,ds\\&=\int_0^1\left[\frac{1}{6}x_1(s)\left(1+\cos^2x_2(s)\right)+3\sin(x_1(s))^{1/3}+\cos^2s+1\right]ds\\&\ge\int_0^1\left(\frac{1}{6}M-2\right)ds>0.\end{aligned}$$

If $x_1(t)<-M$ holds for any $t\in[0,1]$, then

$$\begin{aligned}F_1(x_1,x_2,e_1,e_2)&=\int_0^1[f_1(s,x(s))+e_1(s)]\,ds\\&=\int_0^1\left[\frac{1}{6}x_1(s)\left(1+\cos^2x_2(s)\right)+3\sin(x_1(s))^{1/3}+\cos^2s+1\right]ds\\&\le\int_0^1\left(-\frac{1}{6}M+5\right)ds<0.\end{aligned}$$

Hence,

$$\sum_{j=1}^3A_j\int_0^{\eta_j}[f(s,x(s))+e(s)]\,ds\neq0,$$

and condition ($H_2$) holds. Taking $M^*=61$, for any $d\in R^2$, when $\|d\|>M^*$, then either $\|d\|=|d_1|>M^*$ or $\|d\|=|d_2|>M^*$.

If $\|d\|=|d_1|>M^*$, then $|d_1|\ge|d_2|$ and

$$\begin{aligned}&d^{\mathsf T}\cdot\left(\sum_{j=1}^3A_j\eta_j\right)^{-1}\sum_{j=1}^3A_j\int_0^{\eta_j}[f(s,d)+e(s)]\,ds\\&=(d_1,d_2)\left(\int_0^1(f_1(s,d)+e_1(s))\,ds,\int_0^1(f_2(s,d)+e_2(s))\,ds\right)^{\mathsf T}\\&=\int_0^1\left[\frac{1}{6}d_1^2\left(1+\cos^2d_2\right)+3d_1\sin(d_1)^{1/3}+\left(\cos^2s+1\right)d_1\right.\\&\quad\left.+\frac{1}{6}d_2^2\left(1+e^{-\sin^2d_1}\right)+3d_2\sin(d_2)^{1/3}+\left(\cos^2s+1\right)d_2\right]ds\\&>\int_0^1\left[\frac{1}{6}d_1^2-10|d_1|\right]ds>0.\end{aligned}$$

If $\|d\| = |d_2| > M^*$, then $|d_2| \geq |d_1|$ and

$$\begin{aligned} & d^\top \cdot \left(\sum_{j=1}^{3} A_j \eta_j\right)^{-1} \sum_{j=1}^{3} A_j \int_0^{\eta_j} [f(s,d)+e(s)]\, ds \\ &= (d_1, d_2)\left(\int_0^1 (f_1(s,d)+e_1(s))\, ds, \int_0^1 (f_2(s,d)+e_2(s))\, ds\right)^\top \\ &= \int_0^1 \left[\frac{1}{6} d_1^2 \left(1+\cos^2 d_2\right) + 3d_1 \sin(d_1)^{1/3} + \left(\cos^2 s + 1\right) d_1 \right. \\ &\quad \left. + \frac{1}{6} d_2^2 \left(1 + e^{-\sin^2 d_1}\right) + 3d_2 \sin(d_2)^{1/3} + \left(\cos^2 s + 1\right) d_2 \right] ds \\ &> \int_0^1 \left[\frac{1}{6} d_2^2 - 10|d_2|\right] ds > 0. \end{aligned}$$

So condition ($H_3$) holds. Hence, from Theorem 2.1, BVP (2.20),(2.21) has at least one solution $x \in C([0,1], R^2)$.

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