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The optimal harvesting problems of a stage-structured population

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Abstract

In this paper, we consider a stage-structured population model with two life stages, immature and mature. The optimal harvesting policy for the considered population is made. We obtain the optimal harvesting effort, the maximum sustainable yield, the maximum sustainable economic rent and the corresponding population density.

Keywords: Optimal harvesting; Stage structure; The maximum sustainable yield; The maximum sustainable economic rent

1. Introduction

Biological resources are renewable resources. Economic and biological aspects of renewable resources management have been considered by Clark [1]. In recent years, the optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors [2–5]. But in the natural world, there are many species whose individual members have a life history that takes them through two stages, immature and mature. Previously, the stage-structured model of population growth consisting of immature and mature individuals was analyzed in the literature [6–8]. The stage-structured model is described by the equations

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$$\begin{cases} \dot{x} = \beta_1 y - rx - \beta_2 x - \eta_1 x^2, \\ \dot{y} = \beta_2 x - r_2 y - \eta_2 y^2. \end{cases} \tag{1.1}$$

where β_1, β_2, r, r_2, η_1, η_2 are positive constants, x, y represent the immature and mature populations sizes respectively to model stage-structured population growth. The birth rate of the immature population is β_1. The death rate of the immature and the mature are r, r_2 respectively. The immature's transformation rate of mature is β_2. The immature and mature population are all density restriction ($\eta_1 x^2$ and $\eta_2 y^2$).

Generally speaking, the exploitation of population should be determined by the economic and biological value of the population. It is the purpose of this paper to analyze the exploitation of the stage-structured model. In Section 2, the basic model with harvesting is given, we analyze the existence and stability of equilibria. In Section 3, we consider the optimal harvesting policy for the immature and the mature respectively. we choose the maximum sustainable yield as the management objective and consider the optimal harvesting policy for the considered population. Explicit expression are obtained for the optimal harvesting effort, the maximum sustainable yield and the corresponding population level. In Section 3.1, we deal with the problem which the immature and the mature are harvested simultaneously. we choose the maximum sustainable economic rent as the management objective and consider the optimal harvesting policy for the considered population. Explicit expression are obtained for the optimal harvesting effort, the maximum sustainable economic rent and the corresponding population level.

2. The existence and stability of the equilibria

In this section, we analyze the model with harvesting. We use the phrase catch-per-unit-effort hypothesis [1] to describe an assumption that the harvest is proportional to the stock level, or that $h_1 x$ or $h_2 y$, where $h_1 \geqslant 0$, $h_2 \geqslant 0$ are constants denoting the harvesting efforts about the immature and the mature respectively. For convenience in mathematics, without loss of generality, we always assume that $r_1 = r + \beta_2$ in all that follows. We set up the harvesting model

$$\begin{cases} \dot{x} = \beta_1 y - r_1 x - \eta_1 x^2 - h_1 x = P(x, y), \\ \dot{y} = \beta_2 x - r_2 y - \eta_2 y^2 - h_2 y = Q(x, y). \end{cases} \tag{2.1}$$

Considering the biological significance, we study system (2.1) in the region

$$D = \{(x, y) \in R^2 : x \geqslant 0, y \geqslant 0\}.$$

Now we carry out existence and stability of the equilibria of (2.1), which are solutions of

$$\begin{cases} \beta_1 y - r_1 x - \eta_1 x^2 - h_1 x = 0, \\ \beta_2 x - r_2 y - \eta_2 y^2 - h_2 y = 0. \end{cases} \tag{2.2}$$

We are only interested in the non-negative equilibria.

Theorem 2.1. $(0,0)$ *is always an equilibrium of system* (2.1).

(1) *If* $\beta_1\beta_2 < (r_1 + h_1)(r_2 + h_2)$, *then* $(0,0)$ *is a stable node.*
(2) *If* $\beta_1\beta_2 > (r_1 + h_1)(r_2 + h_2)$, *then* $(0,0)$ *is a saddle point.*

Proof. Obviously, $(0,0)$ is an equilibrium system (2.1). The Jacobian matrix corresponding to the linearized system of (2.1) is

$$J(x,y) = \begin{pmatrix} -r_1 - h_1 - 2\eta_1 x & \beta_1 \\ \beta_2 & -r_2 - h_2 - 2\eta_2 y \end{pmatrix}.$$

For the equilibrium $(0,0)$, Jacobian is reduces to

$$J(x,y) = \begin{pmatrix} -r_1 - h_1 & \beta_1 \\ \beta_2 & -r_2 - h_2 \end{pmatrix}.$$

Hence, the stability of $(0,0)$ determined by the characteristic equation's eigenvalues

$$(\lambda + r_1 + h_1)(\lambda + r_2 + h_2) - \beta_1\beta_2 = 0.$$

Solving it produces

$$\lambda_1 + \lambda_2 < 0$$

and

$$\lambda_1\lambda_2 = (r_1 + h_1)(r_2 + h_2) - \beta_1\beta_2.$$

We have

$$\begin{aligned} \Delta &= [r_1 + h_1 + r_2 + h_2]^2 - 4(r_1 + h_1)(r_2 + h_2) + 4\beta_1\beta_2 \\ &= [(r_1 + h_1) - (r_2 + h_2)]^2 + 4\beta_1\beta_2 > 0. \end{aligned}$$

Hence, when (1) is satisfied, $(0,0)$ is a stable node. When (2) is satisfied, $(0,0)$ is a saddle point. This completes the proof. □

The positive equilibrium of system (2.1) is the intersection of the isoclines

$$\begin{cases} l_1 : \beta_1 y - r_1 x - \eta_1 x^2 - h_1 x = 0, \\ l_2 : \beta_2 x - r_2 y - \eta_2 y^2 - h_2 y = 0. \end{cases} \tag{2.3}$$

Obviously, the graph of l_1 and l_2 are parabolas. l_1 is symmetric to line $x = -(r_1 + h_1)/2\eta_1 < 0$ and l_2 is symmetric $y = -(r_2 + h_2)/2\eta_2 < 0$. We denote

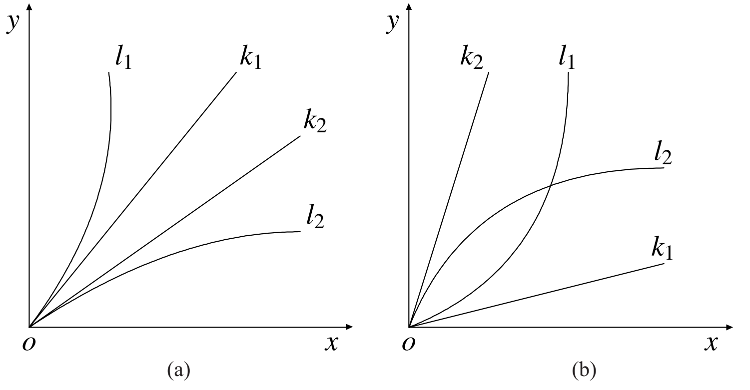

Fig. 1.

the intersections by $(x_0,\ y_0)$. We interested in the existence and stability of positive equilibria of (2.1).

Suppose that k_i $(i = 1, 2)$ denotes the slope of the tangent line of l_i $(i = 1, 2)$ at $(0, 0)$. Clearly $k_1 = (r_1 + h_1)/\beta_1,\ k_2 = \beta_2/(r_2 + h_2)$ and $0 < k_1,\ 0 < k_2$. If $k_1 \geqslant k_2$, the curves l_1, l_2 do not intersect in the positive quadrant, which implies that the positive equilibrium (x_0, y_0) vanishes. That is to say, the unique non-negative equilibrium is $(0, 0)$ (see Fig. 1(a)). If $k_1 < k_2$, the intersection in the positive quadrant is unique, which is the unique positive equilibrium (x_0, y_0) (see Fig. 1(b)).

Theorem 2.2. *If* $\beta_1\beta_2 > (r_1 + h_1)(r_2 + h_2)$, *then there exists a unique positive equilibrium* (x_0, y_0) *of* (2.1), *which is a stable node. If* $\beta_1\beta_2 < (r_1 + h_1)(r_2 + h_2)$, *then there exists no positive equilibrium,* $(0, 0)$ *is the unique equilibrium in the positive quadrant.*

Proof. From the above discussion, the existence of the positive equilibrium is over. Now, we analyze the local geometric properties of (x_0, y_0). The Jacobian matrix of (x_0, y_0) is

$$J(x_0, y_0) = \begin{pmatrix} -r_1 - h_1 - 2\eta_1 x_0 & \beta_1 \\ \beta_2 & -r_2 - h_2 - 2\eta_2 y_0 \end{pmatrix}.$$

For the Jacobian also can be reduced to

$$J(x_0, y_0) = \begin{pmatrix} -\frac{\beta_1 y_0}{x_0} - \eta_1 x_0 & \beta_1 \\ \beta_2 & -\frac{\beta_2 x_0}{y_0} - \eta_2 y_0 \end{pmatrix}.$$

Hence, the stability of (x_0, y_0) determined by the characteristic equation's eigenvalues

$$\lambda^2 + \left[\frac{\beta_1 y_0}{x_0} + \eta_1 x_0 + \frac{\beta_2 x_0}{y_0} + \eta_2 y_0\right]\lambda + \left(\frac{\beta_1 y_0}{x_0} + \eta_1 x_E\right)\left(\frac{\beta_1 x_0}{y_0} + \eta_2 y_0\right)$$
$$- \beta_1\beta_2 = 0.$$

Solving it produces $\lambda_1 + \lambda_2 < 0$, $\lambda_1\lambda_2 > 0$. We have

$$\Delta = \left[\frac{\beta_1 y_0}{x_0} + \eta_1 x_0 + \frac{\beta_2 x_0}{y_0} + \eta_2 y_0\right]^2 - 4\left(\frac{\beta_1 y_0}{x_0} + \eta_1 x_0\right)\left(\frac{\beta_2 x_0}{y_0} + \eta_2 y_0\right)$$
$$+ 4\beta_1\beta_2$$
$$= \left[\left(\frac{\beta_1 y_0}{x_0} + \eta_1 x_0\right) - \left(\frac{\beta_2 x_0}{y_0} + \eta_2 y_0\right)\right]^2 + 4\beta_1\beta_2 > 0.$$

Hence, $\lambda_1 < 0$, $\lambda_2 < 0$. (x_0, y_0) is a stable node. The proof is completed. □

Theorem 2.3. *Each trajectory of* (2.1) *starting in D is positive-going bounded.*

Proof. We want to construct an outer boundary of a positive invariant region which contains (x_0, y_0). Let AB and BC be the line segments of $L_1 : x = p$, $L_2 : y = q$, and (p, q) is any fixed point in D satisfying $p > x_0$ and

$$\frac{-(r_2 + h_2) + \sqrt{(r_2 + h_2)^2 + 4\beta_2\eta_2 p}}{2\eta_2} < q < \frac{(r_1 + h_1)p + \eta_1 p^2}{\beta_1}.$$

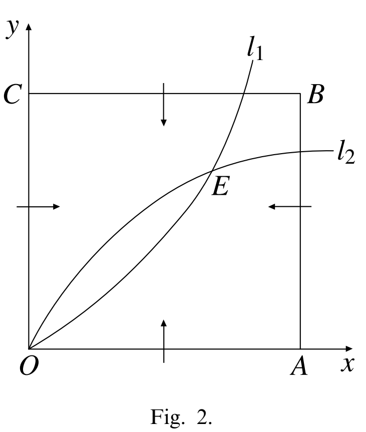

Fig. 2.

The domain enclosed by OABCO, where A$(p,0)$, B(p,q), C$(0,q)$ (see Fig. 2). By the sign of $\dot{x}$, $\dot{y}$, we can say that the trajectory starting from (p,q) of (2.1) cannot leave the confined set.

This completes the proof. □

Theorem 2.4. *If $\beta_1\beta_2 > (r_1+h_1)(r_2+h_2)$, then the unique positive equilibrium (x_0,y_0) is globally asymptotically stable.*

Proof. By Theorem 2.4, we can easily prove that in D

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} = -r_1-h_1-\eta_1 x-(r_2+h_2)-\eta_2 y<0$$

then by Poincare–Bendixon theorem there are no limit cycles in D, and (x_0,y_0) is the unique positive equilibrium which is stable node in D, so it is globally asymptotically stable. This completes the proof. □

Theorem 2.5. *If $\beta_1\beta_2 < (r_1+h_1)(r_2+h_2)$, then the unique non-negative equilibrium $(0,0)$ is globally asymptotically stable.*

Proof. We construct the following Liapunov function

$$V(x,y)=\beta_2 x+(r_1+h_1)y.$$

Calculating the derivative of $V(x,y)$ along (2.1), we have

$$V'_{(2.1)}(x,y)=[\beta_1\beta_2-(r_1+h_1)(r_2+h_2)]y-\beta_2\eta_1 x^2-(r_1+h_1)\eta_2 y^2 \leqslant 0.$$

We can see that in the domain D, $V'_{(2.1)}=0$ if and only if $x=0$, $y=0$. Hence $(0,0)$ is globally asymptotically stable. This completes the proof. □

The biological significance of Theorems 2.4 and 2.5 is: if the harvesting effort is big enough, such that $\beta_1\beta_2 < (r_1+h_1)(r_2+h_2)$, then the system (2.1) is doom to extinct whenever the initial value is, and vise versa. That is to say, if $\beta_1\beta_2 > (r_1+h_1)(r_2+h_2)$, then the system is persistence and non-oscillation.

3. The optimal harvesting policy of harvesting respectively

In this section, we study the optimal harvesting policy of system (2.1) when the immature and mature are harvested respectively. We choose the maximum sustainable yield as the management objective and consider the optimal harvesting policy for the considered population. Explicit expression are obtained for the optimal harvesting effort, the maximum sustainable yield and the corresponding population level.

3.1. The optimal harvesting policy of the immature

In real life, the harvesting policy is determined by the economic and biological value, if the economic value of the immature population is very high, people will only harvest the immature. In this case, the system (2.1) is

$$\begin{cases} \dot{x} = \beta_1 y - r_1 x - \eta_1 x^2 - h_1 x, \\ \dot{y} = \beta_2 x - r_2 y - \eta_2 y^2. \end{cases} \tag{3.1}$$

We study the optimal harvesting yield of system (3.1). By Theorem 2.5, we can easily proof that the system (3.1) is persistence if and only if $h_1 < ((\beta_1\beta_2)/r_2) - r_1$. The objective function takes the form

$$Y_1 = h_1 x = \beta_1 y - r_1 x - \eta_1 x^2,$$

with the adjoint equation

$$0 = \beta_2 x - r_2 y - \eta_2 y^2,$$

where Y_1 represents the harvesting yield, and

$$Y_1 = \beta_1 y - r_1\left(\frac{r_2 y + \eta_2 y^2}{\beta_2}\right) - \eta_1\left(\frac{r_2 y + \eta_2 y^2}{\beta_2}\right)^2.$$

According to the biological significance of the objective function, we can obtain the optimal population level by letting $\mathrm{d}Y_1/\mathrm{d}t = 0$.

$$y^* = \frac{A}{6\eta_1\eta_2} + \frac{B}{2A\eta_2} - \frac{r_2}{2\eta_2},$$

$$\begin{cases} A = \left(\left(27\beta_1\beta_2^2\eta_2 + \sqrt{\dfrac{27(2\beta_1\eta_1 r_2 - r_2^2)^3 + (27\beta_1\beta_2^2\eta_2)^2\eta_1}{\eta_1}}\right)\eta_1^2\right)^{1/3}, \\ B = \eta_1 r_2^2 - 2\eta_2\beta_2 r_1. \end{cases} \tag{3.2}$$

Hence,

$$\begin{aligned} x^* &= (r_2 y^* + \eta_2 y^{*^2})/(\beta_2) \\ &= (1/6r_2(A^2 + 3\eta_1 B - 3r_2\eta_1 A)/(\eta_1\eta_2 A) \\ &\quad + 1/36(A^2 + 3\eta_1 B - 3r_2\eta_1 A)^2/(\eta_2\eta_1^2 A^2))/\beta_2 \\ &= 1/36(A^2 + 3\eta_1 B - 3r_2\eta_1 A)(3r_2\eta_1 A + A^2 + 3\eta_1 B)/(\eta_2\eta_1^2 A^2\beta_2). \end{aligned}$$

Now we can obtain the explicit expression of the optimal harvesting effort

$$\begin{aligned} h_1^* &= \frac{\beta_1 y^*}{x^*} - r_1 - \eta_1 x^* \\ &= 6\beta_1\eta_1 A\beta_2/(3r_2\eta_1 A + A^2 + 3\eta_1 B) - r_1 \\ &\quad - 1/36(A^2 + 3\eta_1 B - 3r_2\eta_1 A)(3r_2\eta_1 A + A^2 + 3\eta_1 B)/(\eta_1\eta_2 A^2\beta_2). \end{aligned}$$

The maximum sustainable yield is

$$\begin{aligned} Y_1 &= h_1^* x^* \\ &= 1/36(6\beta_1\eta_1 A\beta_2)/(3r_2\eta_1 A + A^2 + 3\eta_1 B) - r_1 \\ &\quad - \frac{(A^2 + 3\eta_1 B - 3r_2\eta_1 A)(3r_2\eta_1 A + A^2 + 3\eta_1 B)}{(\eta_1\eta_2 A^2\beta_2)} \\ &\quad \times \frac{(A^2 + 3\eta_1 B - 3r_2\eta_1 A)(3r_2\eta_1 A + A^2 + 3\eta_1 B)}{\eta_2\eta_1^2 A^2\beta_2}. \end{aligned}$$

3.2. The optimal harvesting policy of the mature

It is the purpose of this paper to analyze the exploitation of the mature population in the stage-structured model. The mature is rather appropriate to the economic and biological views of renewable resources management. The basic model which harvest the mature population is given. In this case, the system (2.1) is

$$\begin{cases} \dot{x} = \beta_1 y - r_1 x - \eta_1 x^2, \\ \dot{y} = \beta_2 x - r_2 y - \eta_2 y^2 - h_2 y. \end{cases} \tag{3.3}$$

We study the optimal harvesting yield of system (3.1). By Theorem 2.5, we can easily proof that the system (3.1) is persistence if and only if $h_2 < ((\beta_1\beta_2)/r_1) - r_2$. The objective function takes the form

$$Y_2 = h_2 y = \beta_2 x - r_2 y - \eta_2 y^2,$$

with the adjoint equation

$$0 = \beta_1 y - r_1 x - \eta_1 x^2.$$

Analyze the function Y_2, according to the biological significance, Let $\mathrm{d}Y_2/\mathrm{d}t = 0$, similar to the analysis of Section 3.1 then we obtain

$$x^* = \frac{C}{6\eta_1\eta_2} + \frac{D}{2C\eta_1} - \frac{r_1}{2\eta_1},$$

$$\begin{cases} C = \left(\left(27\beta_2\beta_1^2\eta_1 + \sqrt{\dfrac{27(2\beta_2\eta_2 r_1 - r_1^2)^3 + (27\beta_2\beta_1^2\eta_1)^2\eta_2}{\eta_2}}\right)\eta_2^2\right)^{1/3}, \\ D = \eta_2 r_1^2 - 2\eta_1\beta_1 r_2. \end{cases} \tag{3.4}$$

Hence,

$$\begin{aligned} y^* &= (r_1x^* + \eta_1 x^{*^2})/(\beta_1) \\ &= (1/6r_1(C^2 + 3\eta_2 D - 3r_1\eta_2 C)/(\eta_1\eta_2 C) \\ &\quad + 1/36(C^2 + 3\eta_2 D - 3r_1\eta_2 C)^2/(\eta_1\eta_2^2 C^2))/\beta_1 \\ &= 1/36(C^2 + 3\eta_2 D - 3r_1\eta_2 C)(3r_1\eta_2 C + C^2 + 3\eta_2 D)/(\eta_1\eta_2^2 C^2\beta_1). \end{aligned}$$

Now we can obtain the explicit expression of the optimal harvesting effort

$$\begin{aligned} h_2^* &= \frac{\beta_2 x^*}{y^*} - r_2 - \eta_2 y^* \\ &= 6\beta_2\eta_2 C\beta_1/(3r_1\eta_2 C + C^2 + 3\eta_2 D) - r_2 \\ &\quad - 1/36(C^2 + 3\eta_2 D - 3r_1\eta_2 C)(3r_1\eta_2 C + C^2 + 3\eta_2 D)/(\eta_1\eta_2 C^2\beta_1). \end{aligned}$$

The maximum sustainable yield is

$$\begin{aligned} Y_2 &= h_2^* y^* \\ &= 1/36(6\beta_2\eta_2 C\beta_1)/(3r_1\eta_2 C + C^2 + 3\eta_2 D) - r_2 \\ &\quad - \frac{(C^2 + 3\eta_2 D - 3r_1\eta_2 C)(3r_1\eta_2 C + C^2 + 3\eta_2 D)}{(\eta_2\eta_1 C^2\beta_1)} \\ &\quad \times \frac{(C^2 + 3\eta_2 D - 3r_1\eta_2 C)(3r_1\eta_2 C + C^2 + 3\eta_2 D)}{\eta_1\eta_2^2 C^2\beta_1}. \end{aligned}$$

4. The optimal harvesting policy of harvesting simultaneously

4.1. The optimal harvesting policy without constraint condition

In this section, we study the optimal harvesting policy of system (2.1) when the immature and mature are harvested simultaneously. we choose the maximum sustainable economic rent as the management objective and consider the optimal harvesting policy for the considered population

$$\begin{cases} \dot{x} = \beta_1 y - r_1 x - \eta_1 x^2 - h_1 x, \\ \dot{y} = \beta_2 x - r_2 y - \eta_2 y^2 - h_2 y. \end{cases} \tag{4.1}$$

If we assume the prices of the per unit of harvested bioassays for the immature and the mature are positive constants p_1, p_2, respectively. The costs of a unit of efforts positive constants are c_1, c_2. The sustainable economic rent is

$$R(h_1, h_2) = p_1 h_1 x + p_2 h_2 y - c_1 h_1 - c_2 h_2. \tag{4.2}$$

In this case, there exists the harvesting effort h_1, h_2. In order to gain the maximum sustainable economic rent, that is, gain the maximum value of Eq. (4.2). According to the practical meaning of system, if the prices satisfy

$$\frac{\beta_1}{r_2} > \frac{p_2}{p_1} > \frac{r_1}{\beta_2}, \tag{4.3}$$

$$2\beta_i^2\eta_i - \left(\frac{p_j}{p_i}\right)^3 \eta_j\beta_j^2 - (2\beta_i\eta_i r_j - \eta_j r_i^2)\frac{p_j}{p_i} > 0 \tag{4.4}$$

we have

$$\begin{cases} \dfrac{\partial(R(h_1,h_2))}{\partial x} & = p_1(-r_1 - 2\eta_1 x) + p_2\beta_2, \\ & = 0, \\ \dfrac{\partial(R(h_1,h_2))}{\partial y} & = p_2(-r_2 - 2\eta_2 y) + p_1\beta_1, \\ & = 0. \end{cases} \tag{4.5}$$

Hence, the optimal population level are given by (x^*, y^*), where

$$x^* = \frac{p_2\beta_2 - p_1 r_1}{2p_1\eta_1}, \quad y^* = \frac{p_1\beta_1 - p_2 r_2}{2p_2\eta_2}.$$

The optimal harvesting effort are given by (h_1^*, h_2^*), where

$$h_1^* = \frac{p_1^2\left(2\beta_1^2\eta_1 - \left(\dfrac{p_2}{p_1}\right)^3 \eta_2\beta_2^2 - (2\beta_1\eta_1 r_2 - \eta_2 r_1^2)\dfrac{p_2}{p_1}\right)}{2p_2\eta_2(p_2\beta_2 - p_1 r_1)},$$

and

$$h_2^* = \frac{p_2^2\left(2\beta_2^2\eta_2 - \left(\dfrac{p_1}{p_2}\right)^3 \eta_1\beta_1^2 - (2\beta_2\eta_2 r_1 - \eta_1 r_2^2)\dfrac{p_1}{p_2}\right)}{2p_1\eta_1(p_1\beta_1 - p_2 r_2)}.$$

If the condition (4.2) and (4.3) is right, then we can easily prove that

$$h_1^* > 0, h_2^* > 0.$$

So the maximum sustainable economic rent is

$$\begin{aligned}R(h_1^*, h_2^*) = &\frac{p_1^2\left(2\beta_1^2\eta_1 - \frac{p_2^3\eta_2\beta_2^2}{p_1^3} - (2\beta_1\eta_1r_2 - \eta_2r_1^2)\frac{p_2}{p_1}\right)}{4(p_2\eta_2\eta_1)} \\ &+ \frac{p_2^2\left(2\beta_2^2\eta_2 - \frac{p_1^3\eta_1\beta_1^2}{p_2^3} - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)}{4(p_1\eta_1\eta_2)} \\ &- c_1p_1^2\frac{\left(2\beta_1^2\eta_1 - \frac{p_2^3\eta_2\beta_2^2}{p_1^3} - (2\beta_1\eta_1r_2 - \eta_2r_1^2)\frac{p_2}{p_1}\right)}{2(p_2\eta_2(p_2\beta_2 - p_1r_1))} \\ &- c_2p_2^2\frac{\left(2\beta_2^2\eta_2 - \frac{p_1^3\eta_1\beta_1^2}{p_2^3} - (2\beta_2\eta_2r_1 - \eta_1r_2^2)p_1/p_2\right)}{2(p_1\eta_1(p_1\beta_1 - p_2r_2))}. \end{aligned} \tag{4.6}$$

From Theorem 2.4, we know that if $\beta_1\beta_2 > (r_1 + h_1)(r_2 + h_2)$, then the system (2.1) is permanent; from Theorem 2.5, $\beta_1\beta_2 < (r_1 + h_1)(r_2 + h_2)$, then the system (2.1) is extinctive.

4.2. *The optimal harvesting policy with constraint condition*

(i) $h_1 = qh$, $h_2 = h$

In this case, we have the harvesting effort for the immature is proportional to the mature's with proportionality constant q. Then the objective function is

$$R(h) = p_1qhx + p_2hy - c(qh + h).$$

Similar to Section 4.1, we have the same conditions and the same optimal population level

$$h^* = \frac{p_2^2\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)}{2p_1\eta_1(p_1\beta_1 - p_2r_2)},$$

the maximum economic rent is

$$\begin{aligned}R(h^*) = &\left(q\frac{p_2\beta_2 - p_1r_1}{2\eta_1} + \frac{p_1\beta_1 - p_2r_2}{2\eta_2} - c(1+q)\right) \\ &\times \frac{p_2^2\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)}{2p_1\eta_1(p_1\beta_1 - p_2r_2)}.\end{aligned}$$

(ii) $h_1 = qh$, $h_2 = h$

In this case, the proportionality q is a variable coefficient. Then the objective function is

$$R(h) = p_1qhx + p_2hy - c(qh + h).$$

Similar to Section 4.1, we have the same conditions and the same optimal population level.

$$h^* = \frac{p_2^2\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2 r_1 - \eta_1 r_2^2)\frac{p_1}{p_2}\right)}{2p_1\eta_1(p_1\beta_1 - p_2 r_2)}$$

and

$$q^* = \frac{p_1^3\eta_1(p_1\beta_1 - p_2r_2)\left(2\beta_1^2\eta_1 - \left(\frac{p_2}{p_1}\right)^3\eta_2\beta_2^2 - (2\beta_1\eta_1r_2 - \eta_2r_1^2)\frac{p_2}{p_1}\right)}{p_2^3\eta_2(p_2\beta_2 - p_1r_1)\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)},$$

the maximum economic rent is

$$\begin{aligned} R(h^*) = \Bigg(& q\frac{p_2\beta_2 - p_1r_1}{2\eta_1} + \frac{p_1\beta_1 - p_2r_2}{2\eta_2} \\ & - c\left(1 + \frac{p_1^3\eta_1(p_1\beta_1 - p_2r_2)\left(2\beta_1^2\eta_1 - \left(\frac{p_2}{p_1}\right)^3\eta_2\beta_2^2 - (2\beta_1\eta_1r_2 - \eta_2r_1^2)\frac{p_2}{p_1}\right)}{p_2^3\eta_2(p_2\beta_2 - p_1r_1)\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)}\right)\Bigg) \\ & \times \frac{p_2^2\left(2\beta_2^2\eta_2 - \left(\frac{p_1}{p_2}\right)^3\eta_1\beta_1^2 - (2\beta_2\eta_2r_1 - \eta_1r_2^2)\frac{p_1}{p_2}\right)}{2p_1\eta_1(p_1\beta_1 - p_2r_2)}. \end{aligned}$$

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