Inductions and restrictions for stable equivalences of Morita type

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Abstract

In this paper, we present two methods, induction and restriction procedures, to construct new stable equivalences of Morita type. Suppose that a stable equivalence of Morita type between two algebras $A$ and $B$ is defined by a $B$-$A$-bimodule $N$. Then, for any finite admissible set $\Phi$ and any generator $X$ of the $A$-module category, the $\Phi$- Auslander-Yoneda algebras of $X$ and $N \otimes_A X$ are stably equivalent of Morita type. Moreover, under certain conditions, we transfer stable equivalences of Morita type between $A$ and $B$ to ones between $e A e$ and $f B f$, where $e$ and $f$ are idempotent elements in $A$ and $B$, respectively. Consequently, for self-injective algebras $A$ and $B$ over a field without semisimple direct summands, and for any $A$-module $X$ and $B$-module $Y$, if the $\Phi$-Auslander-Yoneda algebras of $A \oplus X$ and $B \oplus Y$ are stably equivalent of Morita type for one finite admissible set $\Phi$, then so are the $\Psi$-Auslander-Yoneda algebras of $A \oplus X$ and $B \oplus Y$ for every finite admissible set $\Psi$. Moreover, two representation-finite algebras over a field without semisimple direct summands are stably equivalent of Morita type if and only if so are their Auslander algebras. As another consequence, we construct an infinite family of algebras of the same dimension and the same dominant dimension such that they are pairwise derived equivalent, but not stably equivalent of Morita type. This answers a question by Thorsten Holm.

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1 Introduction

In the representation theory of algebras and groups, there are three fundamental equivalences: Morita, derived and stable equivalences. Roughly speaking, the first two are induced from tensor products of bimodules or two-sided complexes, thus there is a corresponding Morita theory for each (see [17, 20, 9]), while the last one seems not yet to be well understood in this way, and therefore a Morita theory for stable equivalences is missing. Recently, a special class of stable equivalences, called stable equivalences of Morita type, are introduced by Broué in modular representations of finite groups. They are induced by bimodules, have features of a Morita theory, and are shown to be of great interest in modern representation theory since they preserve many homological and structural invariants of algebras and modules (see, for example, [3, 4, 10, 11, 18, 22, 23]). In order to understand this kind of equivalences, one has to know, first of all, examples and basic properties of stable equivalences of Morita type as many as possible. So, one of crucial questions in the course of studying these equivalences is:

Question: How to construct stable equivalences of Morita type for finite-dimensional algebras?

Up to date, only a few methods using trivial extensions, one-point extensions and endomorphism algebras have been known in [19, 14, 15, 16]. Of course, Rickard’s result that the existence of derived equivalences for self-injective algebras implies the one of stable equivalences of Morita type provides another way to construct stable equivalences of Morita type. This method, however, is no longer true for general finite-dimensional algebras (see [8] for some new advances in this direction). So, a systematical method for constructing stable equivalences of Morita type seems not yet to be available.

In this paper, we shall look for a more general and systematical answer to this question, and present two methods, called induction and restriction procedures, to construct new stable equivalences of Morita type for general finite-dimensional algebras. Here our induction procedure has two flexibilities, one is the choice of generators, and the other is the one of finite admissible sets. Thus this construction provides a large variety of stable equivalences of Morita type.

To state our first main result, let us recall the definition of $\Phi$-Auslander-Yoneda algebras in [7]. Let $A$ be a finite-dimensional algebra and $X$ an $A$-module. Then, for an admissible set $\Phi$ of natural numbers, there is defined an algebra $E_{\Phi}A(X)$, called the $\Phi$-Auslander-Yoneda algebra of $X$ in [7], which is equal to $\bigoplus_{i\in \Phi} \text{Ext}^i_A (X,X)$ as a vector space, and its multiplication is defined in a natural way (see Subsection 2.2 below for details). Our main result for inductions reads as follows:

Theorem 1.1. (The Induction Procedure)

Suppose that $A$ and $B$ are finite-dimensional $k$-algebras over a field $k$. Assume that two bimodules $AM_B$ and $BN_A$ define a stable equivalence of Morita type between $A$ and $B$. Let $X$ be an $A$-module which is a generator for $A$-module category. Then, for any finite admissible set $\Phi$ of natural numbers, there is a stable equivalence of Morita type between $E_{\Phi}A(X)$ and $E_{\Phi}B(N \otimes_A X)$.

Note that if $\Phi = \{0\}$, then the above result was known in [16]. Thus Theorem 1.1 generalizes the main result in [16], and provides much more possibilities for constructing stable equivalences of Morita type through the choices of different $\Phi$. Also, our proof of Theorem 1.1 is different from that in [16].

Next, we shall exploit certain kinds of restrictions to construct stable equivalences of Morita type. Our result along this line is the following theorem.

Theorem 1.2. (The Restriction Procedure)

Suppose that $A$ and $B$ are finite-dimensional $k$-algebras over a field $k$ such that neither $A$ nor $B$ has semisimple direct summands. Further, suppose that $AM_B$ and $BN_A$ are bimodules without projective bimodules as direct summands, and define a stable equivalence of Morita type between $A$ and $B$. If $e^2 = e \in A$ such that $M \otimes_B Ne \in \text{add}(Ae)$, and if $f^2 = f \in B$ such that $\text{add}(Bf) = \text{add}(Ne)$, then the bimodules $eMf$ and $fNe$ define a stable equivalence of Morita type between $eAe$ and $fBf$. Moreover, if we define $\Lambda = \text{End}_{eAe}(eA)$,
\[ \Gamma = \text{End}_{fBf}(fB), \quad N' = \text{Hom}_{fBf}((fB)_\Gamma,fNe \otimes_{eAe}(eA)_\Lambda) \quad \text{and} \quad M' = \text{Hom}_{eAe}((eA)_\Lambda,eMf \otimes_{fBf}(fB)_\Gamma), \]

then \( \Gamma N' \Lambda \) and \( \Lambda M' \Gamma \) define a stable equivalence of Morita type between \( \Lambda \) and \( \Gamma \).

In fact, under the assumptions of Theorem 1.2, we may have a more general formulation, namely, for any finite admissible set \( \Phi \) of natural numbers and for any \( eAe \)-module \( X \), the \( \Phi \)-Auslander-Yoneda algebras of \( eAe \oplus X \) and \( fBf \oplus fNe \otimes_{eAe}X \) are stably equivalent of Morita type. This is a consequence of Theorem 1.1 and Theorem 1.2.

Also, from Theorem 1.1 and Theorem 1.2 we have the following characterization of stable equivalences of Morita type for representation-finite algebras as well as for self-injective algebras.

**Corollary 1.3.** Suppose that \( A \) and \( B \) are finite-dimensional \( k \)-algebras over a field \( k \) such that neither \( A \) nor \( B \) has semisimple direct summands.

1. Assume further that \( A \) and \( B \) are self-injective. Let \( X \) be an \( A \)-module and let \( Y \) be a \( B \)-module. If there is a finite admissible set \( \Phi \) of natural numbers such that \( E^\Phi_A(A \oplus X) \) and \( E^\Phi_B(B \oplus Y) \) are stably equivalent of Morita type, then, for any finite admissible set \( \Psi \) of natural numbers, the algebras \( E^\Psi_A(A \oplus X) \) and \( E^\Psi_B(B \oplus Y) \) are stably equivalent of Morita type.

2. Assume additionally that \( A \) and \( B \) are representation-finite. Then \( A \) and \( B \) are stably equivalent of Morita type if and only if \( \Phi = \Psi \).

Note that the “only if” part of Corollary 1.3(2) follows from [16].

Of course, there are many important classes of algebras which are of the form \( \text{End}_A(A \oplus Y) \) with \( A \) self-injective and \( Y \) an \( A \)-module. For example, Schur algebras or \( q \)-Schur algebras. Thus, as a consequence of Corollary 1.3, we know that the global dimension of \( \text{End}_{k[S_n]}(k[S_n] \oplus \Omega(Y)) \) is finite for \( i \in \mathbb{Z} \), where \( k[S_n] \) is the group algebra of the symmetric group \( S_n \), \( Y \) the direct sum of non-projective indecomposable Young modules, and \( \Omega \) the usual syzygy operator.

As another byproduct of our considerations in this paper, we can construct a family of derived equivalent algebras with certain special properties.

**Corollary 1.4.** Suppose that \( k \) is a field with a non-zero element that is not a root of unity. Then, there is an infinite series of \( k \)-algebras of the same dimension such that they have the same dominant and global dimensions, and are all derived equivalent, but pairwise not stably equivalent of Morita type.

The contents of this paper are organized as follows. In Section 2 we fix notations and prepare some basic facts for our proofs. In Section 3 and Section 4 we prove our main results, Theorem 1.1 and Theorem 1.2 as well as Corollary 1.3(2), respectively. In Section 5 we concentrate our consideration on self-injective algebras, and establish some applications of our main results. In particular, in this section we prove Corollary 1.3(1) and supply a sufficient condition, which is used in Section 6 to verify when two algebras are not stably equivalent of Morita type. In Section 6 we apply our results in the previous sections to Liu-Schulz algebras and give a proof of Corollary 1.4 which answers a question by Thorsten Holm.

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## 2 Preliminaries

In this section, we shall fix some notations, and recall some definitions and basic results which are needed in the proofs of our main results.
2.1 Some conventions and homological facts

Throughout this paper, \( k \) stands for a fixed field. All categories and functors will be \( k \)-categories and \( k \)-functors, respectively. Unless stated otherwise, all algebras considered are finite-dimensional \( k \)-algebras, and all modules are finitely generated left modules.

Let \( C \) be a category. Given two morphisms \( f : X \to Y \) and \( g : Y \to Z \) in \( C \), we denote the composition of \( f \) and \( g \) by \( fg \) which is a morphism from \( X \) to \( Z \), while we denote the composition of a functor \( F : C \to D \) between categories \( C \) and \( D \) with a functor \( G : D \to E \) between categories \( D \) and \( E \) by \( GF \) which is a functor from \( C \) to \( E \).

If \( C \) is an additive category and \( X \) is an object in \( C \), we denote by \( \text{add}(X) \) the full subcategory of \( C \) consisting of all direct summands of direct sums of finitely many copies of \( X \). The object \( X \) is called an additive generator for \( C \) if \( \text{add}(X) = C \).

Let \( A \) be an algebra. We denote by \( A\text{-mod} \) the category of all \( A \)-modules, by \( A\text{-proj} \) (respectively, \( A\text{-inj} \)) the full subcategory of \( A\text{-mod} \) consisting of projective (respectively, injective) modules, by \( D \) the usual \( k \)-duality \( \text{Hom}_k(-, k) \), and by \( v_A \) the Nakayama functor \( D\text{Hom}_A(-, A) \) of \( A \). Note that \( v_A \) is an equivalence from \( A\text{-proj} \) to \( A\text{-inj} \) with the inverse \( \text{Hom}_A(D(A), -) \). We denote the global and dominant dimensions of \( A \) by \( \text{gl.dim}(A) \) and \( \text{dom.dim}(A) \), respectively.

As usual, by \( \mathcal{D}^b(A) \) we denote the bounded derived category of complexes over \( A\text{-mod} \). It is known that \( A\text{-mod} \) is fully embedded in \( \mathcal{D}^b(A) \) and that \( \text{Hom}_{\mathcal{D}^b(A)}(X,Y[i]) \cong \text{Ext}^i_A(X,Y) \) for all \( i \geq 0 \) and all \( A\text{-modules} \) \( X \) and \( Y \).

Let \( A \) be an \( A\text{-module} \). We denote by \( \Omega_A^i(X) \) the \( i \)-th syzygy, by \( \text{soc}(X) \) the socle, and by \( \text{rad}(X) \) the Jacobson radical of \( X \).

Let \( X \) be an additive generator for \( A\text{-mod} \). The endomorphism algebra of \( X \) is called the Auslander algebra of \( A \). This algebra is, up to Morita equivalence, uniquely determined by \( A \). Note that Auslander algebras can be described by two homological properties: An algebra \( A \) is an Auslander algebra if \( \text{gl.dim}(A) \leq 2 \leq \text{dom.dim}(A) \).

An \( A\text{-module} \) \( X \) is called a generator for \( A\text{-mod} \) if \( \text{add}(A \cdot X) \subseteq \text{add}(X) \); a cogenerator for \( A\text{-mod} \) if \( \text{add}(D(A \cdot X)) \subseteq \text{add}(X) \); and a generator-cogenerator if it is both a generator and a cogenerator for \( A\text{-mod} \). Clearly, an additive generator for \( A\text{-mod} \) is a generator-cogenerator for \( A\text{-mod} \). But the converse is not true in general.

Let \( T \) be an arbitrary \( A\text{-module} \), and let \( B := \text{End}(A) \) be the endomorphism algebra of \( T \). We consider the following full subcategories of \( A\text{-mod} \) related to \( T \).

\[
\begin{align*}
\text{Gen}(A,T) & := \{ X \in A\text{-mod} \mid \text{there is a surjective homomorphism from } T^m \text{ to } X \text{ with } m \geq 1 \} \\
\text{Pre}(A,T) & := \{ X \in A\text{-mod} \mid \text{there is an exact sequence } T_1 \to T_0 \to X \text{ with all } T_i \in \text{add}(A\cdot T) \} \\
\text{App}(A,T) & := \{ X \in A\text{-mod} \mid \text{there is a homomorphism } g : T_0 \to X \text{ with } T_0 \in \text{add}(A\cdot T) \text{ such that } \text{Ker}(g) \in \text{Gen}(A\cdot T) \text{ and } \text{Hom}_A(T',g) \text{ is surjective for } T' \in \text{add}(T) \}
\end{align*}
\]

The following lemma is known, for a proof, we refer, for example, to [24 Lemma 2.1].

**Lemma 2.1.** Let \( T \) be an \( A\text{-module} \) and \( B = \text{End}(A\cdot T) \). Let \( X \) be an arbitrary \( A\text{-module} \). Then:

1. If \( Y \) is a right \( B\text{-module} \), then the natural homomorphism \( \delta : Y \otimes_B \text{Hom}_A(T,X) \to \text{Hom}_B(\text{Hom}_A(X,T),Y) \), given by \( y \otimes f \mapsto \delta_{\otimes f} \) with \( \delta_{\otimes f}(g) = y(fg) \) for \( y \in Y, f \in \text{Hom}_A(T,X), g \in \text{Hom}_A(X,T) \), is an isomorphism if \( x \in \text{add}(A\cdot T) \).
2. If \( X' \in \text{add}(A\cdot T) \), or \( X \in \text{add}(A\cdot T) \), then the composition map \( \mu : \text{Hom}_A(X',T) \otimes_B \text{Hom}_A(T,X) \to \text{Hom}_A(X',X) \), given by \( f \otimes \otimes g \mapsto fg \) is bijective.
3. If \( X \in \text{Gen}(A\cdot T) \), then the evaluation map \( e_X : T \otimes_B \text{Hom}_A(T,X) \to X \) is surjective. If \( X \in \text{App}(A\cdot T) \), then \( e_X \) is bijective. Conversely, if \( e_X \) is bijective, then \( X \in \text{App}(A\cdot T) \).

The next lemma is taken from [22 Lemma 2.1], which can also be verified directly.
Lemma 2.2. \([22]\) (1) Let \(A, B, C\) and \(E\) be \(k\)-algebras, and let \(A X_B\) and \(B Y_E\) be bimodules with \(X_B\) projective. Put \(X^* = \text{Hom}_B(X, B)\). Then the natural homomorphism \(\Phi : A X \otimes_B Y \to \text{Hom}_B(B X_A^*, B Y_E)\), defined by \(x \otimes y \mapsto \Phi_{x \otimes y}\), where \(\Phi_{x \otimes y}(f) = (xf) y\) for \(x \in X, y \in Y\) and \(f \in X^*\), is an isomorphism of \(A\)-\(E\)-bimodules, where the image of \(x\) under \(f\) is denoted by \(xf\).

(2) In the situation \((E P_A, C X_B, A U_B)\), if \(P_A\) is projective, or if \(X_B\) is projective, then \(E P \otimes_A \text{Hom}_B(C X_B, A U_B) \cong \text{Hom}_B(C X_B, E P \otimes_A U_B)\) as \(E\)-\(C\)-bimodules. Dually, in the situation \((A P_E, B X_C, B U_A)\), if \(A P_E\) is projective, or if \(B X_C\) is projective, then \(\text{Hom}_B(B X_C, B U_A) \otimes_A P_E \cong \text{Hom}_B(B X_C, B U \otimes_A P_E)\) as \(C\)-\(E\)-bimodules.

The following is a well-known result due to Auslander (for example, see \([2]\) Proposition 5.6, p.214).

Lemma 2.3. Let \(\Lambda\) be an Artin algebra such that \(\text{gl.dim}(\Lambda) \leq 2 \leq \text{dom.dim}(\Lambda)\). Let \(U\) be a \(\Lambda\)-module such that \(\text{add}(U)\) is the full subcategory of \(\Lambda\)-\(\text{mod}\) consisting of all projective-injective \(\Lambda\)-modules. Then

1. \(\Lambda := \text{End}(U)\) is representation-finite.
2. \(\Lambda\) is Morita equivalent to \(\text{End}(X)^{op}\), where \(X\) is an additive generator for \(\Lambda\)-\(\text{mod}\).

Finally, we recall the definition of \(\mathcal{D}\)-split sequences from \([6]\). For our purpose, we just restrict our attention to module categories.

Let \(\mathcal{D}\) be a full subcategory of \(A\)-\(\text{mod}\). A short exact sequence

\[
0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0
\]

in \(A\)-\(\text{mod}\) is called a \(\mathcal{D}\)-split sequence if \(M \in \mathcal{D}\), \(\text{Hom}_{\Lambda}(D', g)\) and \(\text{Hom}_{\Lambda}(f, D')\) are surjective for every object \(D' \in \mathcal{D}\).

Note that \(\mathcal{D}\)-split sequences were used in \([6]\) to construct tilting modules of projective dimension at most one.

2.2 Admissible sets and perforated orbit categories

In \([7]\), a class of algebras, called \(\Phi\)-Auslander-Yoneda algebras, were introduced, which include, for example, Auslander algebras, generalized Yoneda algebras and certain trivial extensions.

Let \(\mathbb{N}\) be the set of natural numbers \(\{0, 1, 2, \cdots\}\). Recall that a subset \(\Phi\) of \(\mathbb{N}\) is said to be admissible provided that \(0 \in \Phi\) and that for any \(p, q, r \in \Phi\) with \(p + q + r \in \Phi\) we have \(p + q \in \Phi\) if and only if \(q + r \in \Phi\).

As shown in \([7]\), there are a lot of admissible subsets of \(\mathbb{N}\). For example, given any subset \(S\) of \(\mathbb{N}\) containing 0, the set \(\{x^m \mid x \in S\}\) is admissible for all \(m \geq 3\).

Let \(\Phi\) be an admissible subset of \(\mathbb{N}\).

Let \(\mathcal{C}\) be a \(k\)-category, and let \(F\) be an additive functor from \(\mathcal{C}\) to itself. The \((F, \Phi)\)-orbit category \(C^{F, \Phi}\) of \(\mathcal{C}\) is a category in which the objects are the same as that of \(\mathcal{C}\), and the morphism set between two objects \(X\) and \(Y\) is defined to be

\[
\text{Hom}_{C^{F, \Phi}}(X, Y) := \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{C}}(X, F^i Y) \in k\text{-Mod},
\]

and the composition is defined in an obvious way. Since \(\Phi\) is admissible, \(C^{F, \Phi}\) is an admissible \(k\)-category. In particular, \(\text{Hom}_{C^{F, \Phi}}(X, X)\) is a \(k\)-algebra (which may not be finite-dimensional), and \(\text{Hom}_{C^{F, \Phi}}(X, Y)\) is an \(\text{Hom}_{C^{F, \Phi}}(X, X)\)-\(\text{Hom}_{C^{F, \Phi}}(Y, Y)\)-bimodule. For more details, we refer the reader to \([7]\). In this paper, the category \(C^{F, \Phi}\) is simply called a perforated orbit category, and the algebra \(\text{Hom}_{C^{F, \Phi}}(X, X)\) is called the perforated Yoneda algebra of \(X\) without mentioning \(F\) and \(\Phi\).

In case \(\mathcal{C}\) is the bounded derived category \(\mathcal{D}^b(A)\) with \(A\) a \(k\)-algebra, and \(F\) is the shift functor \([1]\) of \(\mathcal{D}^b(A)\), we denote simply by \(E_A^{\Phi}\) the \((F, \Phi)\)-orbit category \(C^{F, \Phi}\), by \(E_A^{\Phi}(X, Y)\) the set \(\text{Hom}_{E_A^{\Phi}}(X, Y)\), and by \(E_A^{\Phi}(X)\) the endomorphism algebra \(\text{Hom}_{E_A^{\Phi}}(X, X)\) of \(X\) in \(E_A^{\Phi}\). It is called \(\Phi\)-Auslander-Yoneda algebra of \(X\).
Note that each element in \( E^\Phi_A(X, Y) \) can be written as \( (f_i)_{i \in \Phi} \) with \( f_j \in \text{Hom}_{\mathcal{D}^b(A)}(X, Y[j]) \). The composition of morphisms in \( \mathcal{E}^\Phi_A \) can be interpreted as follows: for each triple \((X, Y, Z)\) of objects in \( \mathcal{D}^b(A) \),

\[
E^\Phi_A(X, Y) \times E^\Phi_A(Y, Z) \longrightarrow E^\Phi_A(X, Z)
\]

\[
((f_u)_{u \in \Phi}, (g_v)_{v \in \Phi}) \mapsto (h_i)_{i \in \Phi},
\]

where

\[
h_i := \sum_{u+v=i} f_u(g_v[u])
\]

for each \( i \in \Phi \). Clearly, if \( \Phi \) is finite, then \( E^\Phi_A(X, Y) \) is finite-dimensional for all \( X, Y \in A\text{-mod} \).

Now, let us state some elementary properties of the Hom-functor \( E^\Phi_A(X, -) \).

**Lemma 2.4.** Suppose that \( A \) is an algebra, that \( X \) is an \( A\)-module, and that \( \Phi \) is a finite admissible subset of \( \mathbb{N} \).

1. Let \( \text{add}^\Phi_A(X) \) stand for the full subcategory of \( \mathcal{E}^\Phi_A \) consisting of objects in \( \text{add}(A X) \). Then the Hom-functor \( E^\Phi_A(X, -) : \text{add}^\Phi_A(X) \longrightarrow E^\Phi_A(X)\text{-proj} \) is an equivalence of categories;

2. Let \( B \) be a \( k \)-algebra, and let \( P \) be a \( B\)-\( A \)-bimodule such that \( P \) is projective. Then there is a canonical algebra homomorphism \( \alpha_P : E^\Phi_A(X) \longrightarrow E^\Phi_B(P \otimes_A X) \) defined by \((f_i)_{i \in \Phi} \mapsto (P \otimes_A f_i)_{i \in \Phi}\) for \((f_i)_{i \in \Phi} \in E^\Phi_A(X)\). Thus every left (or right) \( E^\Phi_B(P \otimes_A X) \)-module can be regarded as a left (or right) \( E^\Phi_A(X) \)-module via \( \alpha_P \).

The following homological result plays an important role in proving Theorem 1.1.

**Lemma 2.5.** Suppose that \( A, B \) and \( C \) are \( k \)-algebras. Let \( A X \) be a module, and let \( A Y_B \) and \( B PC \) be bimodules with \( \text{Proj} \) projective. Then, for each \( i \geq 0 \), we have \( \text{Ext}^i_A(X, Y_B \otimes_B PC) \simeq \text{Ext}^i_A(X, Y) \otimes_B PC \) as \( \text{C}^{\text{op}} \)-modules. Moreover, for each admissible subset \( \Phi \) of \( \mathbb{N} \), we have \( E^\Phi_A(X, Y_B \otimes_B PC) \simeq E^\Phi_A(X, Y) \otimes_B PC \) as \( E^\Phi_A(X)\text{-C-bimodules} \).

**Proof.** First, let us recall the Yoneda product. Assume that \( U, V \) and \( W \) are \( A \)-modules. Fix a minimal projective resolution \( P^\bullet_U \) of \( A U \):

\[
\cdots \longrightarrow P^n \overset{d_n}{\longrightarrow} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \overset{d_1}{\longrightarrow} P^0 \overset{d_0}{\longrightarrow} U \longrightarrow 0,
\]

with all \( P^i \) projective. If \( g : U \longrightarrow V \) is a homomorphism, then there is a lifting of \( g \), which is a chain map \( g^* : P^\bullet_U \longrightarrow P^\bullet_V \). Then, for each \( i \geq 1 \), we have a short exact sequence \( 0 \longrightarrow \Omega^i_A(U) \overset{\lambda_i}{\longrightarrow} P^{i-1} \overset{h_i}{\longrightarrow} \Omega^{i-1}_A(U) \longrightarrow 0 \), which gives rise to a right exact sequence of \( k \)-modules

\[
\text{Hom}_A(P^{i-1}, V) \overset{(h_i)_*}{\longrightarrow} \text{Hom}_A(\Omega^i_A(U), V) \longrightarrow \text{Ext}^i_A(U, V) \longrightarrow 0.
\]

Hence each element of \( \text{Ext}^i_A(U, V) \) can be regarded as a homomorphism in \( \text{Hom}_A(\Omega^i_A(U), V) \) modulo the subspace of \( \text{Hom}_A(\Omega^i_A(U), V) \) generated by all homomorphisms that factorize through \( \lambda_i \), where \( i \geq 0 \) and \( P^{i-1} := 0 \). In what follows, we denote the image of \( f \in \text{Hom}_A(\Omega^i_A(U), V) \) by \( \overline{f} \in \text{Ext}^i_A(U, V) \).

Given \( i, j \in \mathbb{N} \), \( f_i \in \text{Hom}_A(\Omega^i_A(U), V) \) and \( g_j \in \text{Hom}_A(\Omega^j_A(V), W) \), we know that the Yoneda product \( \mu : \text{Ext}^i_A(U, V) \otimes_k \text{Ext}^j_A(V, W) \longrightarrow \text{Ext}^{i+j}_A(U, W) \) can be presented by \( \overline{f_i} \otimes_k \overline{g_j} \rightarrow \Omega^i_A(f_i)g_j \), where \( \Omega^i_A(f_i) \) is the \( j \)-th term of a lifting of \( f_i \). Note that the Yoneda product is independent of the choice of a lifting of \( f_i \).

By Lemma 2.2, for each \( AW \), there is a natural \( \text{C}^{\text{op}} \)-module isomorphism \( \theta_W : \text{Hom}_A(W, Y) \otimes_B PC \rightarrow \text{Hom}_A(W, Y_B \otimes_B PC) \) defined by \( \theta_W(f \otimes p)(w) = f(w) \otimes p \) for \( f \in \text{Hom}_A(W, Y), p \in P, \) and \( w \in W \). In other words, we have a natural equivalence \( \theta : \text{Hom}_A(-, Y) \otimes_B PC \simeq \text{Hom}_A(-, Y_B \otimes B PC) \) of functors from \( A\text{-mod} \) to \( \text{C}^{\text{op}}\text{-mod} \). Let

\[
\cdots \longrightarrow Q^i \longrightarrow Q^{i-1} \longrightarrow \cdots \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow X \longrightarrow 0
\]
be a minimal projective resolution of $A X$. Then, by definition, we have a right exact sequence of $k$-modules

$$
\text{Hom}_A(Q^{-1}, Y) \longrightarrow \text{Hom}_A(\Omega^i_A(X), Y) \longrightarrow \text{Ext}_A^i(X, Y) \longrightarrow 0.
$$

Since $B P$ is projective, the following diagram is exact and commutative for $i \geq 0$:

$$
\begin{array}{cccccc}
\text{Hom}_A(Q^{-1}, Y) \otimes_B P_C & \longrightarrow & \text{Hom}_A(\Omega^i_A(X), Y) \otimes_B P_C & \longrightarrow & \text{Ext}_A^i(X, Y) \otimes_B P_C & \longrightarrow & 0 \\
\phi_{Q^{-1}} & \downarrow & \phi_{\Omega^i_A(X)} & \downarrow & \phi & \downarrow & 0 \\
\text{Hom}_A(Q^{-1}, Y \otimes_B P_C) & \longrightarrow & \text{Hom}_A(\Omega^i_A(X), Y \otimes_B P_C) & \longrightarrow & \text{Ext}_A^i(X, Y \otimes_B P_C) & \longrightarrow & 0,
\end{array}
$$

where we set $Q^{-1} := 0$. This induces an isomorphism $\varphi_i : \text{Ext}_A^i(X, Y) \otimes_B P_C \rightarrow \text{Ext}_A^i(X, Y \otimes_B P_C)$ defined by $f_i \otimes p \mapsto \theta_{\Omega^i_A(X)}(f_i \otimes p)$, where $f_i \in \text{Hom}_A(\Omega^i_A(X), Y)$ and $p \in P$. Clearly, $\varphi_i$ is a $C^{op}$-homomorphism for each $i \geq 0$. Thus the first part of Lemma 2.5 is proved.

Second, for each admissible subset $\Phi$ of $\mathbb{N}$, we define a map $\varphi_\Phi : E^0_A(X, Y) \otimes_B P_C \rightarrow E^0_A(X, Y \otimes_B P_C)$ by $f_I \otimes p \mapsto (\varphi_i(f_i \otimes p))$, where $p \in P$, and $f_i \in \text{Hom}_A(\Omega^i_A(X), Y)$ with $i \in \Phi$. By the above discussion, we know that $\varphi_\Phi$ is an isomorphism of $C^{op}$-modules. In order to prove that $\varphi_\Phi$ is an isomorphism of $E^0_A(X)$-$C$-bimodules, it suffices to show that $\varphi_\Phi$ is an isomorphism of left $E^0_A(X)$-modules, or equivalently, we have to check that the following diagram commutes for $i, j \in \Phi$ with $i + j \in \Phi$:

$$
\begin{array}{c}
\text{Ext}_A^i(X, X) \otimes_k \text{Ext}_A^j(X, Y) \otimes_B P_C \\
\downarrow \mu \otimes 1 \\
\text{Ext}_A^{i+j}(X, Y) \otimes_B P_C \\
\downarrow \mu \\
\text{Ext}_A^{i+j}(X, Y) \otimes_B P_C,
\end{array}
$$

where $\mu$ is the usual Yoneda product. Let $u \in \text{Hom}_A(\Omega^i_A(X), X)$, $v \in \text{Hom}_A(\Omega^j_A(X), Y)$ and $p \in P$. Then

$$
((1 \otimes \varphi_j) \mu)(u \otimes v \otimes p) = \Omega^i_A(u) \theta_{\Omega^j_A(X)}(v \otimes p) \quad \text{and} \quad ((\mu \otimes 1) \varphi_{i+j})(u \otimes v \otimes p) = \theta_{\Omega^{i+j}_A(X)}((\Omega^i_A(u)v) \otimes p).
$$

By definition, for each $x \in \Omega^{i+j}_A(X)$, we get

$$
(\Omega^i_A(u) \theta_{\Omega^j_A(X)}(v \otimes p))(x) = (\Omega^i_A(u)v)(x) \otimes p = (\theta_{\Omega^{i+j}_A(X)}((\Omega^i_A(u)v) \otimes p))(x).
$$

It follows that $(1 \otimes \varphi_j) \mu = (\mu \otimes 1) \varphi_{i+j}$. This implies that $\varphi_\Phi$ is an isomorphism of $E^0_A(X)$-$C$-bimodules. Thus the proof is completed. $\square$

## 3 Inductions for stable equivalences of Morita type

In this section, we shall prove Theorem 1.1. First, we recall the definition of stable equivalences of Morita type in [3].

**Definition 3.1.** Let $A$ and $B$ be (arbitrary) $k$-algebras. We say that $A$ and $B$ are stably equivalent of Morita type if there is an $A$-$B$-bimodule $\lambda M_B$ and a $B$-$A$-bimodule $\beta N_A$ such that

1. $M$ and $N$ are projective as one-sided modules, and
2. $M \otimes_B N \simeq A \oplus P$ as $A$-$A$-bimodules for some projective $A$-$A$-bimodule $P$, and $N \otimes_A M \simeq B \oplus Q$ as $B$-$B$-bimodules for some projective $B$-$B$-bimodule $Q$. 


In this case, we say that $M$ and $N$ define a stable equivalence of Morita type between $A$ and $B$. Moreover, we have two exact functors $T_N := N \otimes_A - : A\text{-mod} \to B\text{-mod}$ and $T_M := M \otimes_B - : B\text{-mod} \to A\text{-mod}$. Similarly, the bimodules $P$ and $Q$ define two exact functors $T_P$ and $T_Q$, respectively. Note that the images of $T_P$ and $T_Q$ consist of projective modules.

From now on, we assume that $A, B, M, N, P$ and $Q$ are fixed as in Definition 3.1 and that $X$ is a generator for $A\text{-mod}$. Moreover, we fix a finite admissible subset $\Phi$ of $\mathbb{N}$, and define $\Lambda := E_A^\Phi(N \otimes_A X)$ and $\Gamma := E_B^\Phi(N \otimes_A X)$.

Since the functors $T_N$ and $T_M$ are exact, they preserve acyclicity, and can be extended to triangle functors $T_N' : \mathcal{D}^b(A) \to \mathcal{D}^b(B)$ and $T_M' : \mathcal{D}^b(B) \to \mathcal{D}^b(A)$, respectively. Furthermore, $T_N'$ and $T_M'$ induce canonically two functors $F : \mathcal{E}_A^\Phi \to \mathcal{E}_B^\Phi$ and $G : \mathcal{E}_B^\Phi \to \mathcal{E}_A^\Phi$, respectively. More precisely, if $X^* \in \mathcal{D}^b(A)$, then $F(X^*) := (N \otimes_A X^i, N \otimes_A d_X^i)$, and if $f := (f_j)_{j \in \Phi} \in \text{Hom}_{\mathcal{E}_A^\Phi}(X^*, Y^*)$ with $Y^* \in \mathcal{D}^b(A)$, then $F(f) := (N \otimes_A f_j)_{j \in \Phi} \in \text{Hom}_{\mathcal{E}_B^\Phi}(F(X^*), F(Y^*))$. Similarly, we define the functor $G$.

The functor $F$ gives rise to a canonical algebra homomorphism $\alpha_N : E_A^\Phi(X^*) \to E_B^\Phi(F(X^*))$ for each object $X^* \in \mathcal{D}^b(A)$. In particular, for any $Z^* \in \mathcal{D}^b(B)$, we can regard $E_B^\Phi(Z^*, F(X^*))$ as an $E_B^\Phi(Z^*)$-$E_A^\Phi(X^*)$-bimodule via $\alpha_N$. Note that the homomorphism $\alpha_N$ coincides with the one defined in Lemma 2.4 when $X^*$ is an $A$-module.

**Proof of Theorem 3.1.** We define $U := E_A^\Phi(X, T_M(N \otimes_A X))$ and $V := E_B^\Phi(N \otimes_A X, T_N(X))$. In the following, we shall prove that $U$ and $V$ define a stable equivalence of Morita type between $\Lambda$ and $\Gamma$.

First, we endow $U$ with a right $\Gamma$-module structure by $u \cdot \gamma := uG(\gamma)$ for $u \in U$ and $\gamma \in \Gamma$, and endow $V$ with a right $\Lambda$-module structure by $v \cdot \lambda := vF(\lambda)$ for $v \in V$ and $\lambda \in \Lambda$. Then, $U$ becomes a $\Lambda$-$\Gamma$-bimodule, and $V$ becomes a $\Gamma$-$\Lambda$-bimodule.

By definition, we know $V = \Gamma$, and it is a projective left $\Gamma$-module. Note that $AX$ is a generator and the images of $T_P$ consists of projective modules. We conclude that $T_M(N \otimes_A X) = M \otimes_B (N \otimes_A X) \simeq X \oplus P \otimes_A X \in \text{add}(X)$. Thus $U$ is projective as a left $\Lambda$-module by Lemma 2.4.

(1) $U \otimes_{\Gamma} V$, as a $\Lambda$-$\Lambda$-bimodule, satisfies the condition (2) in Definition 3.1.

Indeed, we define $W := E_A^\Phi(X, (T_M T_N)(X))$, and define a right $\Lambda$-module structure on $W$ by $w \cdot \lambda' := w(GF)(\lambda')$ for $w \in W$ and $\lambda' \in \Lambda$. Then $W$ becomes a $\Lambda$-$\Lambda$-bimodule. Note that there is a natural $\Lambda$-module isomorphism $\varphi : U \otimes_{\Gamma} V \to W$ defined by $x \otimes y \mapsto xG(y)$ for $x \in U$ and $y \in V$. We claim that $\varphi$ is an isomorphism of $\Lambda$-$\Lambda$-bimodules. In fact, it suffices to show that $\varphi$ respects the structure of right $\Lambda$-modules. However, this follows immediately from a verification: for $c \in U, d \in V$ and $a \in \Lambda$, we have

$$\varphi((c \otimes d) \cdot a) = \varphi(c \otimes (dF(a))) = cG(dF(a)) = cG(d)(GF)(a) = \varphi(c \otimes d) \cdot a.$$ Combining this bimodule isomorphism $\varphi$ with Lemma 2.4, we get the following isomorphisms of $\Lambda$-$\Lambda$-bimodules:

$$U \otimes_{\Gamma} V \simeq E_A^\Phi(X, (T_M T_N)(X)) \simeq E_A^\Phi(X, X) \oplus E_A^\Phi(X, P \otimes_A X) = \Lambda \oplus E_A^\Phi(X, P \otimes_A X),$$

where the second isomorphism follows from $M \otimes_B N \simeq A \oplus P$ as $A$-$A$-bimodules, and where the right $\Lambda$-module structure on $E_A^\Phi(X, P \otimes_A X)$ is induced by the canonical algebra homomorphism $\Lambda \to E_A^\Phi(P \otimes_A X)$, which sends $(f_j)_{j \in \Phi}$ in $\Lambda$ to $(P \otimes_A f_j)_{j \in \Phi}$ (see Lemma 2.4(2)).

Now, we show that $E_A^\Phi(X, P \otimes_A X)$ is a projective $\Lambda$-$\Lambda$-bimodule. In fact, since $P \in \text{add}(A \otimes_k A)$, we conclude that $E_A^\Phi(X, P \otimes_A X) \in \text{add}(E_A^\Phi(X, (A \otimes_k A) \otimes_A X))$. Thus, it is sufficient to prove that $E_A^\Phi(X, (A \otimes_k A) \otimes_A X)$ is a projective $\Lambda$-$\Lambda$-bimodule. For this purpose, we first note that the right $\Lambda$-module structure on $E_A^\Phi(X, (A \otimes_k A) \otimes_A X)$ is induced by the canonical algebra homomorphism $\Lambda \to E_A^\Phi((A \otimes_k A) \otimes_A X)$, which sends $g := (g_i)_{i \in \Phi}$ in $\Lambda$ to $(A \otimes_k A) \otimes_A g_i$ in $\Lambda$. Clearly, $(A \otimes_k A) \otimes_A X \in \text{add}(A)$. It follows that $\text{Ext}_{\Lambda}^j((A \otimes_k A) \otimes_A X, (A \otimes_k A) \otimes_A X) = 0$ for any $j > 0$, and therefore $(A \otimes_k A) \otimes_A g_i = 0$ for any $0 \neq i \in \Phi$. Thus we have $\alpha_{A \otimes_k A}(g) = (A \otimes_k A) \otimes_A g_0$.

If $\pi : \Lambda \to \text{End}_A(X)$ is the canonical projection and $\mu'$ is the
canonical algebra homomorphism \( \text{End}_A(X) \rightarrow \text{End}_A((A \otimes_k A) \otimes_A X) \), then \( \alpha_{A \otimes_k A} = \pi \mu' \). Thus the right \( \Lambda \)-module structure on \( E^\Phi_A(X, (A \otimes_k A) \otimes_A X) \) is induced by \( \text{End}_A(X) \). Similarly, from the homomorphisms

\[
\Lambda = E^\Phi_A(X) \xrightarrow{\phi} \text{End}_A(X) \xrightarrow{\mu} \text{End}_A(A \otimes_k X) = E^\Phi_A(A \otimes_k X),
\]

where \( \mu : \text{End}_A(X) \rightarrow \text{End}_A(A \otimes_k X) \) is induced by the tensor functor \( A \otimes_k - \), we see that the right \( \Lambda \)-module structure on \( E^\Phi_A(X, A \otimes_k X) \) is also induced by \( \text{End}_A(X) \). Thus \( E^\Phi_A(X, (A \otimes_k A) \otimes_k X) \simeq E^\Phi_A(X, A \otimes_k X) \) as \( \Lambda \)-\( \Lambda \)-bimodules. Moreover, it follows from Lemma 2.5 that \( E^\Phi_A(X, A \otimes_k X) \simeq E^\Phi_A(X, A) \otimes_k X \) as \( \Lambda \)-\( \Lambda \)-bimodule. Since the \( \Lambda \)-module \( X \) can be regarded as a right \( \Lambda \)-module via the homomorphism \( \pi \), we see that \( X \) is actually isomorphic to \( E^\Phi_A(X, A) \) as right \( \Lambda \)-modules. Thus \( E^\Phi_A(X, A) \otimes_k X \simeq E^\Phi_A(X, A) \otimes_k X \) as \( \Lambda \)-\( \Lambda \)-bimodules. Since \( A \in \text{add}(X) \), we know that \( E^\Phi_A(X, A) \) is a projective \( \Lambda \)-module and \( E^\Phi_A(A, X) \) is a projective right \( \Lambda \)-module. Hence \( E^\Phi_A(X, A) \otimes_k X \) is a projective \( \Lambda \)-\( \Lambda \)-bimodule. This implies that \( E^\Phi_A(X, P \otimes_A X) \) is a projective \( \Lambda \)-\( \Lambda \)-bimodule.

(2) \( V \otimes \Lambda U \), as a \( \Gamma \)-\( \Gamma \)-bimodule, fulfills the condition (2) in Definition 3.1.

Let \( Z := E^B_B(N \otimes_A X, T_NM(N \otimes_A X)) \). Similarly, we endow \( Z \) with a right \( \Gamma \)-module structure defined by \( z \cdot b := z(F) \{(1, 1, 1) \} \) for \( z \in Z \) and \( b \in \Gamma \). Then \( Z \) becomes a \( \Gamma \)-\( \Gamma \)-bimodule. Observe that, for each \( A \)-module \( Y \), there is a homomorphism \( \Psi_Y : V \otimes \Lambda E^B_B(X, Y) \rightarrow E^B_B(N \otimes_A X, T_NY) \) of \( \Gamma \)-modules, which is defined by \( g \otimes h \mapsto g \otimes h \) for \( g \in V \) and \( h \in \Psi_E(X, Y) \). This homomorphism is natural in \( Y \). In other words, \( \Psi_Y \) is an \( \otimes \)-isomorphism of \( \Gamma \)-modules. It follows from \( T_M(N \otimes_A X) \) \( \in \text{add}(X) \) that \( \Psi_{T_M(N \otimes_A X)} : V \otimes \Lambda U \rightarrow Z \) is a \( \otimes \)-isomorphism. Similarly, we can check that \( \Psi_{T_M(N \otimes_A X)} \) preserves the structure of right \( \Gamma \)-modules. Thus \( V \otimes \Lambda U \rightarrow Z \) is an \( \otimes \)-isomorphism of \( \Gamma \)-\( \Gamma \)-bimodules, and there are the following isomorphisms of \( \Gamma \)-\( \Gamma \)-bimodules:

\[ (** ) \quad V \otimes \Lambda U \simeq Z \simeq \Gamma \otimes B \otimes B (N \otimes_A X, Q \otimes B (N \otimes_A X)), \]

where the second isomorphism is deduced from \( N \otimes_A M \simeq B \otimes Q \) as \( B \otimes B \)-bimodules. By an argument similar to that in the proof of (1), we can show that \( E^B_B(N \otimes_A X, Q \otimes B (N \otimes_A X)) \) is a projective \( \Gamma \)-\( \Gamma \)-bimodule.

It remains to show that \( U_T \) and \( V_A \) are \( \Lambda \)-\( \Lambda \)-bimodule. This is equivalent to showing that the tensor functors \( T_U := U \otimes \Gamma - : \text{Mod} \rightarrow \Lambda \)-\( \text{Mod} \) and \( T_V := V \otimes \Lambda - : \text{Mod} \rightarrow \Gamma \)-\( \text{Mod} \) are exact. Since tensor functors are always right exact, the exactness of \( T_U \) is equivalent to the property that \( T_U \) preserves injective homomorphisms of modules. Now, suppose that \( f : C \rightarrow D \) is an injective homomorphism between \( \Gamma \)-modules \( C \) and \( D \). Since \( E^B_B(N \otimes_A X, Q \otimes B (N \otimes_A X)) \) is a right projective \( \Gamma \)-module, we know from (**) that the composition functor \( T_U T_U \) is exact. In particular, the homomorphism \( (T_U(f)) : (T_U T_U)(C) \rightarrow (T_U T_U)(D) \) is injective. Let \( \mu : \text{Ker} (T_U(f)) \rightarrow T_U(C) \) be the canonical inclusion. Clearly, we have \( \mu (\mu (f)) = 0 \), which shows \( T_U (f) = f \) is injective. Hence \( T_U \) preserves injective homomorphisms. Similarly, we can show that \( T_V \) preserves injective homomorphisms, too. Consequently, \( U_T \) and \( V_A \) are \( \Lambda \)-\( \Lambda \)-bimodule.

Thus, the \( \Lambda \)-\( \Lambda \)-bimodules \( U \) and \( V \) define a stable equivalence of Morita type between \( \Lambda \) and \( \Gamma \). This finishes the proof of Theorem 1.1. □

Remarks. (1) If we take \( \Phi = \{ 0 \} \) in Theorem 1.1 then we get [16 Theorem 1.1]. If we assume that \( A \) is a self-injective algebra, then we get a stable equivalence of Morita type between \( E^\Phi_A(A \otimes X) \) and \( E^\Phi_A(A \otimes \Omega_A \otimes X) \) for any \( A \)-module \( X \), any finite admissible subset \( \Phi \) of \( \mathbb{N} \), and any integer \( i \in \mathbb{Z} \). This follows from Theorem 1.1 and the fact that \( \Omega_A \) provides a stable equivalence of Morita type between \( A \) and itself if \( A \) is self-injective. Thus we re-obtain the stable equivalence of [7 Corollary 3.14].

(2) Since stable equivalences of Morita type preserve the global, dominant and finitistic dimensions of algebras, Theorem 1.1 asserts actually also that these dimensions are equal for algebras \( E^\Phi_A(X) \) and \( E^\Phi_B(N \otimes_A X) \).

Many important classes of algebras are of the form \( \text{End}_A(A \otimes X) \) with \( A \) a self-injective algebra. From the above remarks (see also [7 Corollary 3.14]), we may get a series of algebras which are stably equivalent.
of Morita type to Schur algebras. For unexplained terminology in the next corollary, we refer the reader to [5].

**Corollary 3.2.** Suppose that \( k \) is an algebraically closed field. Let \( S_n \) be the symmetric group of degree \( n \). We denote by \( Y \) the direct sum of all non-projective Young modules over the group algebra \( k[S_n] \) of \( S_n \). Then,

1. for every finite admissible subset \( \Phi \) of \( \mathbb{N} \), the algebras \( E_{k[S_n]}^\Phi(k[S_n] \oplus Y) \) and \( E_{k[S_n]}^\Phi(k[S_n] \oplus \Omega^i(Y)) \) are stably equivalent of Morita type for all \( i \in \mathbb{Z} \).

2. All algebras \( \text{End}_{k[S_n]}^i(k[S_n] \oplus \Omega^i(Y)) \) are stably equivalent of Morita type to the Schur algebra \( S_k(n,n) \).

In particular, \( \text{gl.dim}(\text{End}_{k[S_n]}^i(k[S_n] \oplus \Omega^i(Y))) < \infty \) for all \( i \in \mathbb{Z} \).

### 4 Restrictions for stable equivalences of Morita type

In this section, we shall consider the general question of how to transfer stable equivalences of Morita type between algebras \( A \) and \( B \) over a field to the ones between \( eAe \) and \( fBf \), where \( e \) and \( f \) are idempotent elements in \( A \) and \( B \), respectively. In particular, we shall prove Theorem 1.2 in this section.

Before we start with our proof of Theorem 1.2, we state the following facts, which are essentially known in literature. However, we would like to collect them together as a lemma for the convenience of the reader.

**Lemma 4.1.** Suppose that \( A \) and \( B \) are \( k \)-algebras without semisimple direct summands. Assume that \( _AM_B \) and \( _BN_A \) define a stable equivalence of Morita type between \( A \) and \( B \), and that \( M \) and \( N \) do not have any projective bimodules as direct summands. Then,

1. there are isomorphisms of bimodule: \( N \simeq \text{Hom}_A(M,A) \simeq \text{Hom}_B(M,B) \) and \( M \simeq \text{Hom}_A(N,A) \simeq \text{Hom}_B(N,B) \).
2. Both \( (N \otimes_A - , M \otimes_B -) \) and \( (M \otimes_B - , N \otimes_A -) \) are adjoint pairs of functors.
3. There are isomorphisms of bimodules: \( P \simeq \text{Hom}_A(P,A) \) and \( Q \simeq \text{Hom}_B(Q,B) \), where \( P \) and \( Q \) are bimodules defined in Definition 3.7. Moreover, the bimodules \( _AP_A \) and \( _BQ_B \) are projective-injective.
4. If \( IA \) is injective, then so is \( N \otimes_A I \).

**Proof.** (1) Note that, if \( M \) and \( N \) are indecomposable bimodules, then all the statements in Lemma 4.1 have been proved in [4] Theorem 2.7, Corollary 3.1, Lemma 3.2] under the hypothesis of separability on the semisimple quotient algebras \( A/\text{rad}(A) \) and \( B/\text{rad}(B) \). One can check that they are still valid without the hypothesis of separability condition. In the following, we shall use [13] Theorem 2.2] to show Lemma 4.1 under the weaker assumption that \( M \) and \( N \) do not have any projective bimodules as direct summands.

Since \( A \) and \( B \) are stably equivalent of Morita type and do not have any semisimple direct summands, it follows from [13] Proposition 2.1] that \( A \) and \( B \) have the same number of indecomposable direct summands (of two-sided ideals). Suppose that \( A = A_1 \times A_2 \times \cdots \times A_n \) and \( B = B_1 \times B_2 \times \cdots \times B_n \), where all \( A_i \) and all \( B_i \) themselves are indecomposable algebras. By the proof of [13] Theorem 2.2], we know that, up to suitable reordering, for each \( 1 \leq i \leq n \), there is an \( A_i-B_i \)-bimodule \( M_i \) and a \( B_i-A_i \)-bimodule \( N_i \) such that \( M_i \) and \( N_i \) are direct summands of \( M \) and \( N \) as bimodules, respectively, and that \( M_i \) and \( N_i \) define a stable equivalence of Morita type between \( A_i \) and \( B_i \). Set \( M' := \bigoplus_{1 \leq i \leq n} M_i \) and \( N' := \bigoplus_{1 \leq i \leq n} N_i \). Clearly, \( M' \) and \( N' \) are direct summands of \( M \) and \( N \), respectively. Further, one can check directly that \( M' \) and \( N' \) also define a stable equivalence of Morita type between \( A \) and \( B \). Since \( _AM_B \) and \( _BN_A \) do not have any projective bimodules as direct summands, it follows from [15] Lemma 4.8] that \( M \simeq M' \) as \( A-B \)-bimodules and \( N \simeq N' \) as \( B-A \)-bimodules. Note that \( A_i \) and \( B_i \) are indecomposable algebras, and \( M_i \) and \( N_i \) do not have any projective bimodules as direct summands. Then, by [4] Lemma 2.1], we conclude that \( M_i \) and \( N_i \) are indecomposable bimodules. This implies that Lemma 4.1 holds for the algebras \( A_i \) and \( B_i \) together with the bimodules \( M_i \) and \( N_i \) for each \( i \). Consequently, there are isomorphisms of \( B-A \)-bimodules:

\[
\text{Hom}_A(M,A) \simeq \text{Hom}_A(M,\bigoplus_{1 \leq u \leq n} M_u, \bigoplus_{1 \leq v \leq n} A_v) \simeq \bigoplus_{1 \leq u \leq n} \text{Hom}_A(M_u,A_u) \simeq \bigoplus_{1 \leq u \leq n} N_u \simeq N.
\]

Similarly, we can prove another statement in (1).
(2) Note that the pair \((N \otimes_A -, M \otimes_B -)\) is an adjoint pair of functors if and only if \(_A M_B \simeq \text{Hom}_B(N, B)\) as bimodules. Thus (2) is a consequence of (1).

(3) It follows from the proof of \([22, \text{Lemma 4.5}]\) that the first part of (3) holds true, and that \(P\) and \(Q\) are injective as one-sided modules. Furthermore, we claim that \(P\) is an injective bimodule. In fact, it suffices to show that, for any indecomposable direct summand \(P'\) of \(P\), the bimodule \(_A P'_B\) is injective. Since \(\_A P \in \text{add}(\_A \otimes \_k A^\prime \_B)\), there are primitive idempotents \(e_1\) and \(e_2\) of \(A\) such that \(P' \in \text{add}(\_A e_1 \otimes \_k e_2 A)\). This implies that \(_A e_1\) and \(_e_2 A\) are injective modules because \(P'\) is injective as a one-sided module. Thus \(P'\) is an injective bimodule, and so is \(P\). Similarly, we can prove that \(Q\) is injective as a bimodule.

(4) We observe that there is an isomorphism of \(B\)-modules: \(N \otimes_A I \simeq \text{Hom}_A(M, I)\). Since \(M_B\) is projective and \(_A I\) is injective, we see that \(\text{Hom}_A(M, I)\) is an injective \(B\)-module, and so is \(N \otimes_A I\). This completes the proof of Lemma \[4.1\] \(\Box\)

By Lemma \[4.1\] we have the following corollary, which provides examples such that the conditions of Theorem \[1.2\] are satisfied. Note that the last statement in Corollary \[4.2\] below follows also from the proof of \([22, \text{Lemma 4.5}]\).

**Corollary 4.2.** Suppose that \(A\) and \(B\) are \(k\)-algebras. Assume that \(\{e_1, \cdots, e_n\}\) and \(\{f_1, \cdots, f_m\}\) are complete sets of pairwise orthogonal primitive idempotents in \(A\) and in \(B\), respectively. Let \(e\) be the sum of all those \(e_i\) for which \(_A e_i\) is projective-injective, and let \(f\) be the sum of all those \(f_j\) for which \(_B f_j\) is projective-injective. If \(M\) and \(N\) are indecomposable bimodules that define a stable equivalence of Morita type between \(A\) and \(B\), then \(Ne \simeq N \otimes_A Ae \in \text{add}(Bf), Mf \simeq M \otimes_B Bf \in \text{add}(Ae),\) and \(Pe \in \text{add}(Ae)\).

**Proof of Theorem 1.2** Let us remark that if \(A\) and \(B\) have no separable direct summands, then we may assume that \(M\) and \(N\) have no non-zero projective bimodules as direct summands. In fact, If \(M = M' \oplus M''\) and \(N = N' \oplus N''\) such that \(M'\) and \(N'\) have no non-zero projective bimodules as direct summands, and that \(M''\) and \(N''\) are projective bimodules, then it follows from \([15, \text{Lemma 4.8}]\) that \(M'\) and \(N'\) also define a stable equivalence of Morita type between \(A\) and \(B\).

Suppose that \(_A M_B\) and \(_B N_A\) do not have any non-zero projective bimodules as direct summands, and define a stable equivalence of Morita type between \(A\) and \(B\). Then, it follows from Lemma \[4.1(2)\] that \((M \otimes_B -, N \otimes_A -)\) and \((N \otimes_A -, M \otimes_B -)\) are adjoint pairs.

First, we note that \(\text{add}(Ae) = \text{add}(Mf)\) and \(\text{add}(N \otimes_A Mf) = \text{add}(Bf)\). In fact, this follows from the following equalities: \(\text{add}(Ae) = \text{add}(M \otimes_B N \otimes_A Ae) = \text{add}(M \otimes_B Bf) = \text{add}(Mf)\), and the fact that \(\text{add}(N \otimes_A X) = \text{add}(N \otimes_A \text{add}(X))\) for any \(A\)-module \(X\).

Thus, if a statement for the idempotent element \(e\) holds true, then it can be proved similarly for \(f\), and vice versa.

Second, we shall show that the bimodules \(eMf\) and \(fNe\) satisfy the conditions of a stable equivalence of Morita type between \(eAe\) and \(fBf\).

(1) \(fNe\) is projective as an \(fBf\)-module and as a right \(eAe\)-module, respectively.

In fact, we have \(fNe \simeq \text{Hom}_B(Bf, BNe)\) as \(fBf\)-\(eAe\)-bimodules. Since \(Ne \in \text{add}(Bf)\) by the definition of \(f\), we see that \(\text{Hom}_B(Bf, Ne)\) is projective as an \(fBf\)-module, that is, \(fNe\) is projective as an \(fBf\)-module. To see that \(fNe\) is a projective right \(eAe\)-module, we notice that \(\text{add}(Mf) = \text{add}(M \otimes_B Bf) = \text{add}(M \otimes_B BNe) = \text{add}(Ae)\), here we use the assumption \(M \otimes_B BNe \in \text{add}(Ae)\). Since \((M \otimes_B -, N \otimes_A -)\) is an adjoint pair, it follows from \(\text{Hom}_B(Bf, BNe) \simeq \text{Hom}_A(M \otimes_B Bf, Ae) \simeq \text{Hom}_A(Mf, Ae)\) that \(fNe\) is projective as a right \(eAe\)-module since \(Mf \in \text{add}(Ae)\). Thus (1) is proved.

(2) \(eMf\) is projective as an \(eAe\)-module and as a right \(fAf\)-module, respectively. The proof of (2) is similar to that of (1), we omit it here.

(3) \(eMf \otimes_B fNe \simeq eAe \otimes ePe\) as bimodules.

Indeed, by the associativity of tensor products, we have the following isomorphisms of \(eAe\)-eAe-
bimodules:

\[ eM f \otimes_{fBf} fNe \simeq eM \otimes_B Bf \otimes_{fBf} fB \otimes_B Ne \]
\[ \simeq eM \otimes_B Bf \otimes_{fBf} \text{Hom}(Bf, B) \otimes_B Ne \]
\[ \simeq eM \otimes_B Bf \otimes_{fBf} \text{Hom}(Bf, B)Ne \quad (\text{by Lemma } 2.2) \]
\[ \simeq eM \otimes_B Ne \quad (\text{by Lemma } 2.1) \].

Since \( M \) and \( N \) define the stable equivalence of Morita type between \( A \) and \( B \), we have \( M \otimes_B N \simeq A \oplus P \) as \( A \)-\( A \)-bimodules. This implies that \( eM f \otimes_{fBf} fNe \simeq eM \otimes_B Ne \simeq e(A \oplus P)e \simeq eAe \oplus ePe \) as bimodules.

(4) \( ePe \) is a projective \( eAe\)-\( eAe \)-bimodule.

In fact, it suffices to show that, for any indecomposable direct summand \( P' \) of the \( A \)-\( A \)-bimodule \( P \), the \( eAe\)-\( eAe \)-bimodule \( eP'e \) is projective. We assume that \( eP'e \neq 0 \). Since \( P \in \text{add}(A \otimes_k A^{op}) \), there are primitive idempotent elements \( e_1 \) and \( e_2 \) of \( A \) such that \( P' \in \text{add}(Ae_1 \otimes_k e_2A) \). Then \( A\overline{P}e \in \text{add}(Ae_1 \otimes_k e_2A) \subseteq \text{add}(Ae_1) \). This means that \( ePe \) is a direct sum of copies of \( A_1 \). Since \( P'e \in \text{add}(Pe) \subseteq \text{add}(Ae) \), we have \( A_1 \in \text{add}(Ae) \). Consequently, \( eA_1 \) is a projective \( ee \)-module. Now, we show that \( e_2Ae \) is a projective right \( eAe \)-module. Indeed, by Lemma 4.1(1), we have the following isomorphisms of \( A^{op} \)-modules: \( eP \simeq \text{Hom}_A(Ae, P) \simeq \text{Hom}_A(Ae, \text{Hom}_A(P, A)) \simeq \text{Hom}_A(P \otimes A A, A) \simeq \text{Hom}_A(\text{add}(Ae), A) \). This shows that \( eP \in \text{add}(eA) \) since \( e_2Pe \in \text{add}(Ae) \). Thus \( eP' \in \text{add}(eA) \). Since the right \( A \)-module \( eP' \) is a direct sum of copies of \( e_2A \), it follows that \( e_2A \in \text{add}(eA) \) and \( e_2Ae \in \text{add}(eAe) \). Consequently, \( e_2Ae \) is a projective right \( eAe \)-module. Hence \( eA_1 \otimes_k e_2Ae \) is a projective \( eAe\)-\( eAe \)-bimodule, and so is its direct summand \( eP'e \). This shows that \( ePe \) is a projective \( eAe\)-\( eAe \)-bimodule.

(5) Similarly, we can prove that \( fNe \otimes eM f \simeq fBf \oplus fQf \) as bimodules, and that the \( fBf-fBf \)-bimodule \( fQf \) is projective.

Thus, by definition, the bimodules \( eM f \) and \( fNe \) define a stable equivalence of Morita type between \( eAe \) and \( fBf \).

Finally, the last statement of Theorem 1.2 follows from Proposition 4.3 below, which emphasizes the view of functors.

Before we give the formulation of Proposition 4.3, we introduce here a few more notations: Set \( \Lambda = \text{End}_{eAe}(eA) \), \( R = \text{End}_{fBf}(fN) \), \( \Gamma = \text{End}_{fBf}(fB) \), \( N' = \text{Hom}_{fBf}((fB)_{f}, fNe \otimes_{eAe} (eA)_{\Lambda}) \) and \( M' = \text{Hom}_{eAe}((eA)_{\Lambda}, eM f \otimes_{fBf} (fB)_{\Gamma}) \).

Let \( \varphi : A \rightarrow \Lambda \) be the algebra homomorphism defined by sending \( a \in A \) to \( \varphi_a \), where \( \varphi_a : eA \rightarrow eA, ex \mapsto exa \) for \( x \in A \). Similarly, we define an algebra homomorphism \( \psi : B \rightarrow \Gamma \).

Recall that, given a diagram of functors between categories:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
H \downarrow & & \downarrow G \\
C & \xleftarrow{K} & D,
\end{array}
\]

we say that this diagram is commutative if there is a natural isomorphism \( \alpha : GF \rightarrow KH \).

**Proposition 4.3.** (1) The following diagram of functors is commutative

\[
\begin{array}{ccc}
A-\text{mod} & \xrightarrow{N_{\otimes A}} & B-\text{mod} & \xrightarrow{M_{\otimes B}} & A-\text{mod} \\
\downarrow e & & \downarrow f & & \downarrow e \\
eAe-\text{mod} & \xrightarrow{fNe \otimes_{eAe} eA} & fBf-\text{mod} & \xrightarrow{eMe \otimes_{fBf} eA} & eAe-\text{mod}.
\end{array}
\]

In particular, \( fBf fNe \otimes_{eAe} eA \simeq fBf fN \) and \( eAe eM f \otimes_{fBf} fB \simeq_{eAe} eM \).
(2) We have the following commutative diagram of functors

\[
\begin{array}{ccc}
A\text{-mod} & \xrightarrow{N\otimes_A -} & B\text{-mod} & \xrightarrow{M\otimes_B -} & A\text{-mod} \\
\Lambda \otimes_A - & \xrightarrow{\Gamma \otimes_B -} & \Lambda \otimes_A - \\
\Lambda\text{-mod} & \xrightarrow{\Lambda M' \otimes -} & \Lambda\text{-mod}
\end{array}
\]

where the right $A$-module structure on $\Lambda$ and the right $B$-module structure on $\Gamma$ are induced by $\varphi$ and $\psi$, respectively. Moreover, $\tau N'_{\Lambda}$ and $\Lambda M'_{\Gamma}$ define a stable equivalence of Morita type between $\Lambda$ and $\Gamma$.

**Proof.** (1) To prove that the first square in (1) is commutative, it is sufficient to show that there is a natural transformation $\Phi : f Ne \otimes_{eA} e(-) \rightarrow f N \otimes_A -$, which is an isomorphism. Now we define $\Phi$ to be the composition of the following two natural transformations: for each $X \in A$-mod,

\[
\Phi_X : f Ne \otimes_{eA} eX \xrightarrow{\sim} f N \otimes_A A e \otimes_{eA} eX \xrightarrow{id_{fN} \otimes \mu} fN \otimes_A X,
\]

where $\mu : A e \otimes_{eA} eX \rightarrow X$ is the multiplication map. Clearly, we need only to show that $id_{fN} \otimes \mu$ is a natural isomorphism, that is, for each $\lambda X$, we have to show that

\[
fN \otimes_A A e \otimes_{eA} eX \rightarrow fN \otimes_A X
\]

is an isomorphism.

Indeed, we shall first show that if $X \in A$-mod and $eZ = 0$, then $fN \otimes_A Z = 0$. To prove this, we observe that $fN \otimes_A Z \simeq \text{Hom}_B(Bf, N \otimes_A Z) \simeq \text{Hom}_A(A M \otimes_B Bf, Z)$, where the second isomorphism comes from the adjoint pair $(M \otimes_B - , N \otimes_A - )$. Since $\text{add}(Bf) = \text{add}(N \otimes_A A e)$ and $Pe \in \text{add}(A e)$, we have $\text{add}(M \otimes_B Bf) = \text{add}(A e)$. Thus $eZ = 0$ implies that $fN \otimes_A Z = 0$. Next, we consider the exact sequence

\[
0 \rightarrow \text{Ker}(\mu) \rightarrow A e \otimes_{eA} eX \xrightarrow{\mu} X \rightarrow X / AeX \rightarrow 0.
\]

Note that $e\text{Ker}(\mu) = 0 = e(X / AeX)$ and that $fN A \simeq fB \otimes_B N A$ is a projective right $A$-module. By applying tensor functor $f N \otimes_A -$ to the above sequence, we deduce that

\[
fN \otimes_A A e \otimes_{eA} eX \xrightarrow{id_{fN} \otimes \mu} fN \otimes_A X
\]

is an isomorphism. Thus we have proved the commutativity of the left square in (1).

Similarly, we can prove that the right square of $(*)$ commutes. In particular, we see that $f Ne \otimes_{eA} eA \simeq fN A$ as $fBf$-bimodules, and $eMf \otimes_{fBf} fB \simeq eM_B$ as $eAe$-$B$-bimodules.

(2) Note that the bimodules $\text{Hom}_{Ae}(eA, eMf \otimes_{fBf} (fN)_R)$ and $\text{Hom}_{fBf}(fN_R, fNe \otimes_{eA} (eA)_A)$ have been constructed in [16 Theorem 1.1], which induced a stable equivalence of Morita type between $\Lambda$ and $R$. Since $\text{add}(fN) = \text{add}(fB)$, we see that $\text{Hom}_{fBf}(fB, fN)$ and $\text{Hom}_{fBf}(fN, fB)$ induce a Morita equivalence between $R$ and $\Gamma$. As a result, $N'$ and $M'$ define a stable equivalence of Morita type between $\Lambda$ and $\Gamma$. It can be checked directly that $\tau N' \otimes_{\Lambda} \Lambda_A \simeq \Gamma N'_A$ and $\Lambda M' \otimes_{\Gamma} \Gamma_B \simeq \Lambda M'_B$. So, we have

\[
\tau N' \otimes_{\Lambda} \Lambda_A \simeq \Gamma N'_A \simeq \text{Hom}_{fBf}(fB, fN_A) \\
\simeq \text{Hom}_{fBf}(fB, fB \otimes_B N_A) \\
\simeq \tau \Gamma \otimes_B N_A
\]

and

\[
\Lambda M' \otimes_{\Gamma} \Gamma_B \simeq \Lambda M'_B = \text{Hom}_{eA}(eA, eMf \otimes_{fBf} fB_B) \\
\simeq \text{Hom}_{eA}(eA, eM_B) \\
\simeq \text{Hom}_{eA}(eA, eA \otimes_A M_B) \\
\simeq \Lambda A \otimes_A M_B.
\]
This implies that the diagram in (2) is commutative. Thus, we have proved Proposition 4.3. This also finishes the proof of Theorem 1.2. □

Remarks. (1) In Theorem 1.2, the assumption that \( M \) and \( N \) do not have any projective bimodules as direct summands is actually a very mild restriction. In fact, if \( M = X' \oplus X'' \) and \( N = Y' \oplus Y'' \) such that \( X' \) and \( Y' \) have no direct summands of projective bimodules, and that \( X'' \) and \( Y'' \) are projective bimodules, then it follows from [13 Lemma 4.8] that the bimodules \( X' \) and \( Y' \) also define a stable equivalence of Morita type between \( A \) and \( B \). Clearly, we have \( X' \otimes_B Y'e \in \text{add}(Ae) \) and \( (Y'e) \subseteq \text{add}(Ne) \). Since \( _AM \otimes_B Ne \) is a direct summand of \( _AM \otimes_B Ne \), we get \( X' \otimes_B Ne \in \text{add}(Ae) \), and \( Y' \otimes_A X' \otimes_B Ne \in \text{add}(Y' \otimes_A Ae) = \text{add}(Y'e) \). This gives \( Ne \in \text{add}(Y'e) \). Hence \( \text{add}(Y'e) = \text{add}(Ne) \). This means that \( M \) and \( N \) in Theorem 1.2 can be replaced by the bimodules \( X' \) and \( Y' \).

(2) Note that \( M \otimes_B Ne \in \text{add}(Ae) \) is equivalent to \( Pe \in \text{add}(Ae) \). In Theorem 1.2, if \( e \) is an idempotent element in \( A \) such that every indecomposable projective-injective \( A \)-module is isomorphic to a summand of \( Ae \), then \( Pe \in \text{add}(Ae) \). This follows immediately from Lemma 4.1(3).

(3) As pointed out in [4 Section 4], if \( e \) is an idempotent in \( A \) and if \( f \) is an idempotent in \( B \) such that \( \text{add}(Ae) \) and \( \text{add}(Bf) \) are invariant under Nakayama functors, then \( eAe \) and \( fBf \) are self-injective, and any stable equivalence of Morita type between \( A \) and \( B \) induces a stable equivalence of Morita type between \( eAe \) and \( fBf \). Note that we may recover this result from Theorem 1.2 since the idempotents \( e \) and \( f \) satisfy the assumptions of Theorem 1.2 by [4 Lemma 4.1]. In general, however, our algebras \( eAe \) and \( fBf \) in Theorem 1.2 may not be self-injective.

Definition 4.4. [9] Let \( A \) be an algebra. A projective \( A \)-module \( W \) is called a minimal Wedderburn projective module if \( \text{add}(\nu_A(W)) = \text{add}(\nu_0(A) \oplus \nu_1(A)) \), where \( \nu_A \) is the Nakayama functor of \( A \) and \( 0 \rightarrow A \rightarrow I_0(A) \rightarrow I_1(A) \) is the minimal injective copresentation of \( A \). An idempotent element \( e \in A \) is called a minimal Wedderburn idempotent element if \( Ae \) is a minimal Wedderburn projective module.

Auslander proved in [11] that, given \( e^2 = e \in A \), the canonical map \( \rho : A \rightarrow \text{End}_{eAe}(eA) \) defined by right multiplication is an isomorphism if and only if \( \text{add}(Ae) \) contains a minimal Wedderburn projective \( A \)-module.

The following result shows that stable equivalences of Morita type preserve minimal Wedderburn projective modules or minimal Wedderburn idempotent elements.

Lemma 4.5. Suppose that \( A \) and \( B \) are \( k \)-algebras such that \( A \) and \( B \) have no semisimple direct summands. Assume that \( \mu M_B \) and \( \nu N_A \) do not possess any projective bimodules as direct summands, and induce a stable equivalence of Morita type between \( A \) and \( B \). Take a minimal Wedderburn idempotent \( e \in A \) and a minimal Wedderburn idempotent \( f \in B \). Then we have

\[
\text{add}(M \otimes_B Bf) = \text{add}(Ae) \quad \text{and} \quad \text{add}(N \otimes_{A} Ae) = \text{add}(Bf).
\]

Proof. We assume that \( M \otimes_B N \simeq A \oplus P \) as \( A \)-\( A \)-bimodules for some projective \( A \)-\( A \)-bimodule \( P \), and \( N \otimes_A M \simeq B \oplus Q \) as \( B \)-\( B \)-bimodules for some projective \( B \)-\( B \)-bimodule \( Q \). Note that, by Lemma 4.1, the images of the functors \( \alpha P \otimes_A - \) and \( \beta Q \otimes_B - \) consist of projective-injective modules.

Let \( 0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \) and \( 0 \rightarrow B \rightarrow J_0 \rightarrow J_1 \) be minimal injective co-presentations of \( A \) and \( B \), respectively. We claim that

\[
\text{add}(M \otimes_B (J_0 \oplus J_1)) = \text{add}(I_0 \oplus I_1) \quad \text{and} \quad \text{add}(N \otimes_{A} (I_0 \oplus I_1)) = \text{add}(J_0 \oplus J_1).
\]

Clearly, for any \( A \)-module \( X \), we have \( \text{add}(N \otimes_{A} X) = \text{add}(N \otimes_{A} X) \). Since \( 0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \) is exact and \( N_A \) is projective, it follows that

\[
0 \rightarrow B N \rightarrow N \otimes_A I_0 \rightarrow N \otimes_A I_1
\]
is exact. Since \(B \mathcal{N} \otimes_A \mathcal{D} \mathcal{A} \) is injective and \(\text{add}(B \mathcal{B}) = \text{add}(B \mathcal{N})\), we see that \(\text{add}(J_0 \oplus J_1) \subseteq \text{add}(N \otimes_A (I_0 \oplus I_1))\). This implies that \(\text{add}(M \otimes_B (J_0 \oplus J_1)) \subseteq \text{add}(M \otimes_B N \otimes_A (I_0 \oplus I_1))\). Since \(P \otimes_A \mathcal{D} \mathcal{A} \) is projective and injective and since all indecomposable projective-injective \(A\)-modules occur in \(I_0\), we have \(\text{add}(M \otimes_B N \otimes_A (I_0 \oplus I_1)) = \text{add}(I_0 \oplus I_1)\). Thus, \(\text{add}(M \otimes_B (J_0 \oplus J_1)) \subseteq \text{add}(I_0 \oplus I_1)\). Furthermore, it follows from the injectivity of the module \(A M \otimes_B D B\) and \(\text{add}(A A) = \text{add}(A M)\) that \(\text{add}(I_0 \oplus I_1) \subseteq \text{add}(M \otimes_B (J_0 \oplus J_1))\). Thus \(\text{add}(M \otimes_B (J_0 \oplus J_1)) = \text{add}(I_0 \oplus I_1)\). Similarly, we can prove that \(\text{add}(N \otimes_A (I_0 \oplus I_1)) = \text{add}(J_0 \oplus J_1)\). Since \(e \in A\) and \(f \in B\) are minimal Wedderburn idempotents, we see that \(\text{add}(I_0 \oplus I_1) = \text{add}(V_A(A e))\) and \(\text{add}(J_0 \oplus J_1) = \text{add}(V_B(B f))\). Consequently, \(\text{add}(N \otimes_A V_A(A e)) = \text{add}(V_B(B f))\). It follows from \(N \otimes_A V_A(A e) \simeq V_B(N \otimes_A A e)\) that \(\text{add}(V_B(N \otimes_A A e)) = \text{add}(V_B(B f))\). Since the Nakayama functor \(\mathcal{V}_B\) is an equivalence from \(B\)-proj to \(B\)-inj, we deduce that \(\text{add}(N \otimes_A A e) = \text{add}(B f)\). Similarly, we can show that \(\text{add}(M \otimes_B B f) = \text{add}(A e)\). □

In the following we shall see that stable equivalences of Morita type can be transfer to “corner” algebras of Wedderburn type.

**Corollary 4.6.** Suppose that \(A\) and \(B\) are \(k\)-algebras such that \(A\) and \(B\) have no semisimple direct summands. Assume that \(A M_B\) and \(B N_A\) have no projective bimodules as direct summands, and induce a stable equivalence of Morita type between \(A\) and \(B\). Let \(e \in A\) and \(f \in B\) be minimal Wedderburn idempotents. Then \(e M f\) and \(f N e\) define a stable equivalence of Morita type between \(e A e\) and \(f B f\) such that \(e M e \otimes_{e A e} e A e \simeq f N f\) and \(e M f \otimes_{f B f} f B f \simeq e M e\) as bimodules.

**Proof.** By Lemma 4.5 we see that the idempotents \(e\) and \(f\) satisfy the assumptions in Theorem 1.2. Then Corollary 4.6 follows from the first part of Theorem 1.2 together with Proposition 4.3. □

As a corollary of Corollary 4.6, we get the following result.

**Corollary 4.7.** Assume that \(A\) and \(B\) are \(k\)-algebras without semisimple direct summands. Let \(A X\) be a generator-cogenerator for \(A\)-mod, and let \(B Y\) be a generator-cogenerator for \(B\)-mod. If \(\text{End}_A(X)\) and \(\text{End}_B(Y)\) are stably equivalent of Morita type, then there exist bimodules \(A M_B\) and \(B N_A\) which define a stable equivalence of Morita type between \(A\) and \(B\) such that \(\text{add}(A M \otimes_B Y) = \text{add}(A X)\) and \(\text{add}(B N \otimes A X) = \text{add}(B Y)\).

**Proof.** Set \(R = \text{End}_A(X)\) and \(S = \text{End}_B(Y)\). First, we show that if \(A\) does not have any semisimple direct summands, then nor does \(R\).

Suppose contrarily that \(R\) has a semisimple direct summand. Then \(R\) must have a simple projective-injective module \(W\). Since each indecomposable projective-injective \(R\)-module is isomorphic to a direct summand of \(\text{Hom}_A(X, DA)\), there exists an indecomposable projective \(A\)-module \(I\) such that \(W \simeq \text{Hom}_A(X, I)\). Let \(A I\) be the socle of \(A I\). Then \(\text{Hom}_A(X, S)\) can be embedded into the simple \(R\)-module \(\text{Hom}_A(X, I)\), and therefore \(\text{Hom}_A(X, S) \simeq \text{Hom}_A(X, I) \simeq W\) as \(R\)-modules. Since \(A \in \text{add}(X)\), we infer that \(S \simeq I\). Let \(A P\) be the projective cover of \(A S\). Then it follows from \(\text{Hom}_R(\text{Hom}_A(X, P), \text{Hom}_A(X, S)) \simeq \text{Hom}_A(P, S) \neq 0\) that there is a non-zero homomorphism from \(\text{Hom}_A(X, P)\) to the simple projective \(R\)-module \(\text{Hom}_A(X, S)\), which means that \(\text{Hom}_A(X, P) \simeq \text{Hom}_A(X, S)\). Consequently, we get \(P \simeq S \simeq I\). Thus \(A\) has a simple projective-injective module, and therefore it has a semisimple direct summand, which is a contradiction. This shows that \(R\) has no semisimple direct summands. Similarly, we can prove that \(S\) has no semisimple direct summands.

Note that, if \(X\) is a generator-cogenerator for \(A\)-mod, then \(\text{Hom}_A(X, A)\) is a minimal Wedderburn projective \(R\)-module. Similarly, \(\text{Hom}_B(Y, B)\) is a minimal Wedderburn projective \(S\)-module. Clearly, \(\text{End}_R(\text{Hom}_A(X, A)) \simeq A\) and \(\text{End}_S(\text{Hom}_B(Y, B)) \simeq B\). Note that neither \(R\) nor \(S\) has semisimple direct summands. Then, by Corollary 4.6 there exist bimodules \(A M_B\) and \(B N_A\) which define a stable equivalence of Morita type between \(A\) and \(B\). Note that \(\text{Hom}_R(\text{Hom}_A(X, A), R) \simeq A X\) and \(\text{Hom}_S(\text{Hom}_B(Y, B), S) \simeq B Y\). It follows from Corollary 4.6 that \(\text{add}(A M \otimes_B Y) = \text{add}(A X)\) and \(\text{add}(B N \otimes_A X) = \text{add}(B Y)\). □

Combining Corollary 4.7 with [16, Theorem 1.1], we have the following result on Auslander algebras.
Corollary 4.8. Let $A$ and $B$ be representation-finite $k$-algebras. Suppose that $A$ and $B$ have no semisimple direct summands. Let $\Lambda$ and $\Gamma$ be the corresponding Auslander algebras of $A$ and $B$, respectively. Then $\Lambda$ and $\Gamma$ are stably equivalent of Morita type if and only if so are $A$ and $B$.

For an algebra $A$, we denote by $[A]$ the class of all algebras $B$ such that there is a stable equivalence of Morita type between $B$ and $A$. It is known that $[A] = [A \times S]$ for any separable algebra $S$. Note that, if $k$ is a perfect field, then the class of all semisimple $k$-algebras is the same as that of all separable $k$-algebras.

The following result establishes a one-to-one correspondence, up to stable equivalence of Morita type and Auslander algebras. This is an immediate consequence of Corollary 4.8.

Corollary 4.9. Suppose that $k$ is a perfect field. Let $A$ and $B$ be representation-finite $k$-algebras. Suppose that $A$ and $B$ have no semisimple direct summands, then the class of all semisimple $k$-algebras between representation-finite algebras and Auslander algebras. This is an immediate consequence of Corollary 4.8.

Finally, we remark that Corollary 4.8 is not true for derived equivalences. Nevertheless, it was shown in [7] that if two representation-finite, self-injective algebras $A$ and $B$ are derived-equivalent then so are their Auslander algebras. The converse of this statement is open. For further information on constructing derived equivalences, we refer the reader to the current papers [6,7].

5 Stable equivalences of Morita type based on self-injective algebras

Of particular interest are stable equivalences of Morita type between self-injective algebras or between those related to self-injective algebras. Since derived equivalences between self-injective algebras imply stable equivalences of Morita type by a result of Rickard [19], this makes stable equivalences of Morita type closely related to the Broué abelian defect group conjecture which essentially predicates a derived equivalence between two block algebras [3], and thus also a stable equivalence of Morita type between them.

In this section, we will apply Theorem 1.1 and Theorem 1.2 to self-injective algebras. It turns out that the existence of a stable equivalence of Morita type between $\Phi$-Auslander-Yoneda algebras of generators for one finite admissible set $\Phi$ implies the one for all finite admissible sets.

Throughout this section, we fix a finite admissible subset $\Phi$ of $\mathbb{N}$, and assume that $A$ and $B$ are indecomposable, non-simple, self-injective algebras. Let $X$ be a generator for $A$-mod with a decomposition $X := A \oplus \bigoplus_{1 \leq i \leq n} X_i$, where $X_i$ is indecomposable and non-projective such that $X_i \not\cong X_t$ for $1 \leq i \neq t \leq n$, and let $Y$ be a generator for $B$-mod with a decomposition $Y := B \oplus \bigoplus_{1 \leq j \leq m} Y_j$, where $Y_j$ is indecomposable and non-projective such that $Y_j \not\cong Y_s$ for $1 \leq j \neq s \leq m$.

Lemma 5.1. (1) The full subcategory of $E_A^{\Phi}(X)$-mod consisting of projective-injective $E_A^{\Phi}(X)$-modules is equal to $\text{add}(E_A^{\Phi}(X,A))$. Particularly, if $E_A^{\Phi}(X) \neq \text{End}_A(X)$, then $\text{dom.dim}(E_A^{\Phi}(X)) = 0$.

(2) $E_A^{\Phi}(X)$ has no semisimple direct summands.

Proof. (1) For convenience, we set $A_0 = \text{End}_A(X)$ and $\Lambda = E_A^{\Phi}(X)$. Since $A$ is self-injective, it follows from [7, Lemma 3.5] that $v_A(E_A^{\Phi}(X,A)) \simeq E_A(X, v_A A) \simeq E_A^{\Phi}(X, DA) \oplus (E_A^{\Phi}(X,A))$. Consequently, $E_A^{\Phi}(X,A)$ is a projective-injective $\Lambda$-module. We claim that, up to isomorphism, each indecomposable projective-injective $\Lambda$-module is a direct summand of $E_A^{\Phi}(X,A)$. To prove this claim, it suffices to show that $E_A^{\Phi}(X,X_i)$ is not injective for all $1 \leq i \leq n$. We denote $E_A^{\Phi}(X,X_i)$ by $X_i$ for abbreviation.
First, we observe that \( \text{rad}(\Lambda) = \bigoplus_i \text{rad}(\Lambda_i) \oplus \Lambda_+ \), where \( \Lambda_+ = \bigoplus_{i \in \Phi} \Lambda_i \) with \( \Lambda_i = \text{Ext}_A^0(X,X) = \text{Hom}_{\mathcal{O}_1}(X,X^i) \). Since each summand \( \text{Hom}_{\mathcal{O}_1}(X,X^j) \) of \( \tilde{X}_i \) is a \( \mathcal{O}_1 \)-module and since the socle of \( \tilde{X}_i \) is the set of all elements \( x \) in \( \tilde{X}_i \) such that \( \text{rad}(\Lambda)x = 0 \), we see that the socle of \( \tilde{X}_i \) contains \( \bigoplus_{j \in \Phi} \{ x \in \text{soc}_{\Lambda_0}(\text{Ext}_A^i(X,X)) \mid \Lambda_+x = 0 \} \). By an argument of graded modules, we can even see that
\[
\text{soc}_\Lambda(\tilde{X}_i) = \bigoplus_{j \in \Phi} \{ x \in \text{soc}_{\Lambda_0}(\text{Ext}_A^i(X,X)) \mid \Lambda_+x = 0 \}.
\]

Next, we shall show that \( \tilde{X}_m \) is not injective for \( 1 \leq m \leq n \). Indeed, let \( f : X_m \to I \) be an injective envelope of \( X_m \) with \( I \) an injective \( A \)-module. Then \( f_* : \text{Hom}_A(X,X_m) \to \text{Hom}_A(X,I) \) is an injective envelop of the \( \mathcal{O}_1 \)-module \( \text{Hom}_A(X,X_m) \) in \( \mathcal{O}_1 \)-mod. Now, we consider the following two cases:

(a) If \( X_m = \text{Hom}_A(X,X_m) \), then \( X_m \) is annihilated by \( \Lambda_+ \). Since \( X_m \) is not injective in \( \mathcal{O}_1 \)-mod, we conclude that \( \text{Hom}_A(X,X_m) \) is not an injective \( \mathcal{O}_1 \)-module, which implies that \( X_m \) is not injective as a \( \Lambda \)-module.

(b) If \( X_m \neq \text{Hom}_A(X,X_m) \), then there is a positive integer \( t \) such that \( \text{Ext}_A^t(X,X_m) \neq 0 \). We may assume that \( t \) is the maximal number in \( \Phi \) with this property, that is, \( \text{Ext}_A^t(X,X_m) = 0 \) for all \( s \in \Phi \) with \( t < s \). It follows that \( \Lambda_+ \text{Ext}_A^t(X,X_m) = 0 \), which implies that \( 0 \neq \text{soc}_\Lambda(\text{Ext}_A^t(X,X_m)) \subseteq \text{soc}_\Lambda(X_m) \).

Now we consider \( \text{soc}_{\Lambda_0}(\text{Hom}_A(X,X_m)) \). Since \( f_* \) is an injective envelop in \( \mathcal{O}_1 \)-mod, we know that \( \text{soc}_{\Lambda_0}(\text{Hom}_A(X,X_m)) \simeq \text{soc}_{\Lambda_0}(\text{Hom}_A(X,I)) \). Since \( \nu_{\Lambda_0}(\text{Hom}_A(X,A)) \in \text{add}(\text{Hom}_A(X,A)) \) and \( I \in \text{add}(A) \), we see that \( \text{Hom}_A(\text{Hom}_A(X,X_m),\text{soc}_{\Lambda_0}(\text{Hom}_A(X,I))) = 0 \) for \( 1 \leq i \leq n \). If \( e \) is the idempotent in \( \Lambda_0 \) corresponding to the direct summand \( A \) of \( X \), then \( e \text{soc}_{\Lambda_0}(\text{Hom}_A(X,I)) = 0 \). Consequently, \( e \text{soc}_{\Lambda_0}(\text{Hom}_A(X,X_m)) = 0 \). Since \( A \) is self-injective, this implies that \( \text{soc}_{\Lambda_0}(\text{Hom}_A(X,X_m)) \subseteq \text{soc}_\Lambda(X_m) \).

Thus the \( \Lambda \)-submodule \( \text{soc}_{\Lambda_0}(\text{Hom}_A(X,X_m)) \subseteq \text{soc}_\Lambda(X_m) \) is contained in the socle of \( X_m \). This implies that \( X_m \) is not injective since its socle is not simple.

Thus \( \text{add}(E_A^\Phi(X,A)) \) is just the full subcategory of \( E_A^\Phi(X) \)-mod consisting of projective-injective modules.

Finally, we consider the dominant dimension of \( \text{dom.dim}(E_A^\Phi(X)) \). Suppose \( E_A^\Phi(X) \neq \text{End}_A(X) \). Since \( A \) is self-injective, we have \( E_A^\Phi(X,A) = \text{Hom}_A(X,A) \). It follows that \( E_A^\Phi(X,A) \) is annihilated by \( \Lambda_+ \), but not by \( \Lambda \). Hence \( \Lambda \) cannot be cogenerated by \( E_A^\Phi(X,A) \). This implies that \( \text{dom.dim}(E_A^\Phi(X)) = 0 \). We finish the proof.

(2) Contrarily, we suppose that the algebra \( E_A^\Phi(X) \) has a semisimple direct summand. Then \( E_A^\Phi(X) \) has a simple projective-injective module \( S \). According to (1), we know that \( S \) must be a simple projective-injective \( \text{End}_A(X) \)-module. Then it follows from the first part of the proof of Corollary 4.7 that \( A \) has a semisimple direct summand. Clearly, this is contrary to our initial assumption that \( A \) is indecomposable and non-simple.

Thus \( E_A^\Phi(X) \) has no semisimple direct summands. \( \square \)

**Theorem 5.2.** If the algebras \( E_A^\Phi(X) \) and \( E_B^\Psi(Y) \) are stably equivalent of Morita type, then \( n = m \) and there are bimodules \( A_M \otimes_B \otimes_B \) which define a stable equivalence of Morita type between \( A \) and \( B \) such that, up to the ordering of indices, \( A_M \otimes_B \otimes_B \) is a \( A \)-module, where \( A_P \) is projective for all \( i \) with \( l \leq i \leq n \). Moreover, for any finite admissible subset \( \Psi \) of \( \mathbb{N} \), there is a stable equivalence of Morita type between \( E_A^\Phi(X) \) and \( E_B^\Psi(Y) \).

**Proof.** For convenience, we set \( \Lambda_0 = \text{End}_A(X) \), \( \Lambda = E_A^\Phi(X) \), \( \Gamma_0 = \text{End}_B(Y) \) and \( \Gamma = E_B^\Phi(Y) \). By Lemma [5.1] the algebras \( \Lambda \) and \( \Gamma \) have no semisimple direct summands. Let \( e \) be the idempotent in \( \Lambda_0 \) corresponding to the direct summand \( A \) of \( X \), and let \( f \) be the idempotent in \( \Gamma_0 \) corresponding to the direct summand \( B \) of \( Y \). Note that \( e \Lambda \simeq E_A^\Phi(X,A) \) as \( \Lambda \)-modules and \( \Gamma f \simeq E_B^\Phi(Y,B) \) as \( \Gamma \)-modules. Clearly, \( e \Lambda \simeq A \) and \( f \Gamma f \simeq B \) as algebras. Moreover, we see that \( e \Lambda \simeq X \) as \( A \)-modules, and \( f \Gamma \simeq Y \) as \( B \)-modules. Suppose that a stable
equivalences of Morita type between $A$ and $\Gamma$ is given. By Corollary 4.2 and Lemma 5.1, we know that the idempotent $e$ in $A$ and the idempotent $f$ in $\Gamma$ satisfy the conditions in Theorem 1.2. It follows from Theorem 1.2 and Proposition 4.3(i) that there are bimodules $AM_B$ and $BN_A$ which define a stable equivalence of Morita type between $A$ and $B$ such that $\text{add}(M \otimes_B Y) = \text{add}(X)$. By the given decompositions of $X$ and $Y$, we conclude that $n = m$ and, up to the ordering of direct summands, we may assume that $AM \otimes_B Y_i \simeq X_i \oplus P_i$ as $A$-modules, where $AP_i$ is projective for all $i$ with $1 \leq i \leq n$. Now, the last statement in this corollary follows immediately from Theorem 1.1. Thus the proof is completed. $\square$

Usually, it is difficult to decide whether an algebra is not stably equivalent of Morita type to another algebra. The next corollary, however, gives a sufficient condition to assert when two algebras are not stably equivalent of Morita type between $A$ and $\Lambda$.

**Corollary 5.3.** Let $n$ be a non-negative integer. Let $W$ be an indecomposable non-projective $A$-module. Suppose that $\Omega^j_A(W) \not\simeq W$ for any non-zero integer $s$. Set $W_n = \bigoplus_{0 \leq i < n} \Omega^i_A(W)$. Then, for any finite admissible subset $\Psi$ of $\mathbb{N}$, the algebras $E^\Psi_A(A \oplus W_n \oplus \Omega^l_A(W))$ and $E^n_A(A \oplus W_n \oplus \Omega^m_A(W))$ are not stably equivalent of Morita type whenever $m$ and $l$ belong to $\mathbb{N}$ with $n < m < l$.

**Proof.** Suppose that there is a finite admissible subset $\Psi$ of $\mathbb{N}$ such that $E^\Psi_A(A \oplus W_n \oplus \Omega^m_A(W))$ and $E^n_A(A \oplus W_n \oplus \Omega^l_A(W))$ are stably equivalent of Morita type for some fixed $m, l, n \in \mathbb{N}$ with $n < m < l$. Set $\Phi_1 = \{0, 1, \cdots, n\} \cup \{l\}$ and $\Phi_2 = \{0, 1, \cdots, n\} \cup \{m\}$. Then, by Theorem 5.2, we know that there exist bimodules $A \mathcal{M}_4$ and $A \mathcal{N}_4$ which define a stable equivalence of Morita type between $A$ and itself, and that there is a bijection $\sigma : \Phi_1 \rightarrow \Phi_2$ such that $M \otimes_A \Omega^j_A(W) \simeq \Omega^{\sigma(j)}(W) \oplus P_j$ as $A$-modules, where $P_j$ is projective for each $j \in \Phi_1$. In particular, we have $M \otimes_A W \simeq \Omega^{\sigma(0)}(W) \oplus P_0$. Since $M$ is projective as a one-sided module, we know that $M \otimes_A \Omega^j_A(W) \simeq \Omega^{\sigma(0) + l}(W) \oplus P'_l$ with $P'_l \in \text{add}(A_A)$. Note that $M \otimes_A \Omega^j_A(W) \simeq \Omega^{\sigma(l)}(W) \oplus P_l$. It follows that $\Omega^{\sigma(0) + l}(W) \simeq \Omega^{\sigma(l)}(W)$. Consequently, we have $\sigma(l) = \sigma(0) + l \geq l$ since $W$ is not $\Omega$-periodic. Hence $l \leq \sigma(l) \leq m < l$, a contradiction. This shows that $E^\Psi_A(A \oplus W_n \oplus \Omega^m_A(W))$ and $E^n_A(A \oplus W_n \oplus \Omega^l_A(W))$ cannot be stably equivalent of Morita type whenever $l$ and $m \in \mathbb{N}$ with $n < m < l$. $\square$

This corollary will be used in the next section.

6 A family of derived-equivalent algebras: application to Liu-Schulz algebras

In this section, we shall apply our results in the previous sections to solve the following problem on derived equivalences and stable equivalences of Morita type:

**Problem.** Is there any infinite series of finite-dimensional $k$-algebras such that they have the same dimension and are all derived-equivalent, but not stably equivalent of Morita type?

This problem was originally asked by Thorsten Holm at a workshop in Goslar, Germany. Recall that Liu and Schulz in [12] constructed a local symmetric $k$-algebra $A$ of dimension 8 and an indecomposable A-module $M$ such that all the syzygy modules $\Omega^j(M)$ with $n \in \mathbb{Z}$ are 4-dimensional and pairwise non-isomorphic. This algebra $A$ depends on a non-zero parameter $q \in k$, which is not a root of unity, and has an infinite DTi-orbit in which each module has the same dimension. A thorough investigation of Auslander-Reiten components of this algebra was carried out by Ringel in [21]. Based on this symmetric algebra and a recent result in [6] together with the results in the previous sections, we shall construct an infinite family of algebras, which provides a positive solution to the above problem.

From now on, we fix a non-zero element $q$ in the field $k$, and assume that $q$ is not a root of unity. The 8-dimensional $k$-algebra $A$ defined by Liu-Schulz is an associative algebra (with identity) over $k$ with the generators: $x_0, x_1, x_2$, and
the relations: \( x_i^2 = 0, \) and \( x_{i+1}x_i + qx_ix_{i+1} = 0 \) for \( i = 0, 1, 2. \)

Here, and in what follows, the subscript is modulo 3.

Let \( n \) be a fixed natural number, and let \( \Phi = \{0\} \) or \( \{0, 1\} \). For \( j \in \mathbb{Z} \), set \( u_j := x_2 + q^j x_1, I_j := Au_j, J_j := u_jA, I := \bigoplus_{i=0}^n I_i \) and \( \Lambda_j^\Phi := E_j^\Phi(A \oplus I \oplus I_j) \).

With these notations in mind, the main result in this section can be stated as follows:

**Theorem 6.1.** For any \( m \geq n + 4 \), we have

1. \( \dim_k(\Lambda_m^\Phi) = \dim_k(\Lambda_{m+1}^\Phi) \).
2. \( \text{gl.dim}(\Lambda_m^\Phi) = \infty. \)
3. \( \text{dom.dim}(\Lambda_m^\Phi) = \begin{cases} 2 & \text{if } \Phi = \{0\}, \\ 0 & \text{if } \Phi = \{0, 1\}. \end{cases} \)
4. \( \Lambda_m^\Phi \) and \( \Lambda_{m+1}^\Phi \) are derived-equivalent.
5. If \( l > m \), then \( \Lambda_l^\Phi \) and \( \Lambda_m^\Phi \) are not stably equivalent of Morita type.

An immediate consequence of Theorem 6.1 is the following corollary, which solves the above mentioned problem positively.

**Corollary 6.2.** There exists an infinite series of finite-dimensional \( k \)-algebras \( A_i, i \in \mathbb{N} \), such that

1. \( \dim_k(A_i) = \dim_k(A_{i+1}) \) for all \( i \in \mathbb{N} \),
2. all \( A_i \) have the same global and dominant dimensions,
3. all \( A_i \) are derived-equivalent, and
4. \( A_i \) and \( A_j \) are not stably equivalent of Morita type for \( i \neq j \).

The proof of Theorem 6.1 will cover the rest of this section. Let us first introduce a few more notations and conventions.

Let \( B \) be an algebra and \( S \) a subset of \( B \). Set \( R(S) := \{ b \in B \mid sb = 0 \text{ for all } s \in S \} \) for the right annihilator of \( S \) in \( B \), and \( L(S) := \{ b \in B \mid bs = 0 \text{ for all } s \in S \} \) for the left annihilator of \( S \) in \( B \). In case \( x \in B \), we write \( R(x) \) and \( L(x) \) for \( R(\{x\}) \) and \( L(\{x\}) \), respectively. For \( y, z \in B \), we set \( B(y, z) := \{ b \in B \mid L(y)bz = 0 \} \). Note that \( L(S) \) and \( R(S) \) are left and right ideals in \( B \), respectively.

Let \( V \) be a \( k \)-vector space with \( \gamma_i \in V \) for \( 1 \leq i \leq n \in \mathbb{N} \). We denote by \( <\gamma_1, \ldots, \gamma_n> \) the \( k \)-subspace of \( V \) generated by all \( \gamma_i \).

The following result is useful for our calculations, it may be of its own interest in describing the endomorphism rings of direct sums of cyclic left ideals.

**Lemma 6.3.** Let \( B \) be a \( k \)-algebra, and let \( x, y \) and \( z \) be elements in \( B \). Then the following statements hold:

1. There is an isomorphism of \( k \)-vector spaces:
   \[ \varphi_{x,y} : \text{Hom}_k(Bx, By) \xrightarrow{\sim} R(L(x)) \cap By, \]
   which sends \( f \) to \( f(x) \) for \( f \in \text{Hom}_k(Bx, By) \).
2. There is an isomorphism of \( k \)-vector spaces:
   \[ \theta_{x,y} : \text{Hom}_k(Bx, By) \xrightarrow{\sim} B(x,y)/L(y), \]
   which sends \( h \) to \( \overline{h} \) for \( h \in \text{Hom}_k(Bx, By) \), where \( b \in B \) such that \( h(x) = by \) and \( \overline{B} \) stands for the coset \( b + L(y) \).
3. The maps \( \theta_{x,y} \) and \( \theta_{x,z} \) are isomorphisms of algebras.
4. The map \( \theta_{x,y} \) satisfies the following identity:
   \[ \theta_{x,y}(agc) = \theta_{x,y}(a)\theta_{x,y}(g)\theta_{x,y}(c) \]
for \( a \in \text{End}_A(Bx), g \in \text{Hom}_B(Bx, By) \) and \( c \in \text{End}_B(By) 

(5) The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_B(Bx, By) \otimes_{\text{End}_B(By)} \text{Hom}_B(By, Bz) & \xrightarrow{\Delta_{x,y,z}} & \text{Hom}_B(Bx, Bz) \\
\theta_{x,y} \otimes \theta_{y,z} & \simeq & \theta_{x,z}
\end{array}
\]

\[
(B(x,y)/L(y)) \otimes_{\text{End}_B(By)/L(y)} (B(y,z)/L(z)) \xrightarrow{\nabla_{x,y,z}} B(x,z)/L(z),
\]

where \( \Delta_{x,y,z} \) is the composition map, and \( \nabla_{x,y,z} \) is the multiplication map.

(6) Let \( n \) be a positive integer, and let \( x_i \) be elements in \( B \) for \( 1 \leq i \leq n \). We define

\[
M_B(x_1, x_2, \cdots, x_n) := \{ (\overline{t_{i,j}})_{1 \leq i, j \leq n} \mid \overline{t_{i,j}} \in B(x_i, x_j)/L(x_j) \text{ for all } 1 \leq i, j \leq n \}.
\]

Then \( M_B(x_1, x_2, \cdots, x_n) \) becomes an associative \( k \)-algebra with the usual matrix addition and multiplication which is given by \( \nabla_{x_i, x_j, x_k} \). More precisely, there is a natural algebra isomorphism \( \theta : \text{End}_B(\bigoplus_{0 \leq i \leq n} Bx_i) \longrightarrow M_B(x_1, x_2, \cdots, x_n) \), which is induced by \( \theta_{x_i, x_j} \) for \( 1 \leq i, j \leq n \).

**Proof.** (1) Let \( f \in \text{Hom}_B(Bx, By) \). Since \( f \) is a homomorphism of \( B \)-modules, we know \( b(xf) = 0 \) whenever \( b \in B \) and \( bx = 0 \). This implies that \( xf \in R(L(x)) \cap By \). Thus the map \( \varphi_{x,y} \) is well-defined. It is not hard to check that \( \varphi_{x,y} \) is an isomorphism of \( k \)-vector spaces.

(2) For \( x \in B \), we denote by \( \rho_x \) the right multiplication map from \( B \) to itself, defined by \( b \mapsto bx \) for \( b \in B \). Then there is a canonical exact sequence of \( B \)-modules: \( \delta_x : 0 \rightarrow L(x) \xrightarrow{\lambda_x} B \xrightarrow{\pi_x} Bx \rightarrow 0 \), where \( \lambda_x \) is the inclusion, and \( \pi_x \) is the canonical multiplication of \( x \). By the definition of \( B(x,y) \), any element \( w \in B(x,y) \) belongs to \( B(x,y) \) if and only if \( \lambda_x \rho_x \pi_y = 0 \), or equivalently, if and only if \( t \in Bx \) and \( \rho_x \pi_y = 0 \). Clearly, \( w \in L(y) \) if and only if \( \rho_x \pi_y = 0 \). So, we have \( L(y) \subseteq B(x,y) \).

First, we show that \( \theta_{x,y} \) is well-defined. In fact, if \( f \in \text{Hom}_B(Bx, By) \), then there is an element \( b \in B \), which may not be unique, such that the following diagram of left \( B \)-modules commutes:

\[
\begin{array}{cccc}
0 & \xrightarrow{0} & L(x) & \xrightarrow{\lambda_x} B & \xrightarrow{\pi_x} Bx & \xrightarrow{0} \\
\rho_b & \xrightarrow{\rho_b} & \lambda_x & \xrightarrow{f} & \pi_x & \xrightarrow{0}
\end{array}
\]

where \( \rho_b \) is the restriction of \( \rho_b \) to \( L(x) \). Hence \( b \in B(x,y) \). If there is another \( d \in B \) also making the above diagram commutative, then \( (\rho_b - \rho_d) \pi_y = 0 \), and therefore \( \rho_b - \rho_d \) factorizes through \( L(y) \). This implies that \( b - d \in L(y) \) and \( B = Bx \) in \( B(x,y)/L(y) \). Thus \( \theta_{x,y} \) is well-defined.

Next, we shall prove that \( \theta_{x,y} \) is an isomorphism of \( k \)-vector spaces. Indeed, if \( \theta_{x,y}(f) = \overline{0} = \overline{0}_x \) for some map \( f \in \text{Hom}_B(Bx, By) \), then \( b \in L(y) \) and \( \pi_y f = \rho_x \pi_y = 0 \). Since \( \pi_y \) is surjective, we get \( f = 0 \). Thus \( \theta_{x,y} \) is injective. That \( \theta_{x,y} \) is surjective follows from the equivalent definitions of \( B(x,y) \) discussed above.

(3) to (5) follow from the above proof of (2).

(6) is a consequence of (2) to (5). \( \square \)

Recall that, for \( i \in \mathbb{Z} \), we have defined \( u_i := x_2 + q^i x_1, I_i := Au_i \) and \( J_i := u_i A \). In the following lemma, we display a few properties about the Liu-Schulz algebra \( A \).

**Lemma 6.4.** [12][21] (1) The Liu-Schulz algebra \( A \) is an \( \mathbb{N} \)-graded algebra, namely, \( A = \bigoplus_{i \geq 0} A_i \) with \( A_0 = k \), \( A_1 = \langle x_0, x_1, x_2 \rangle \), \( A_2 = \langle x_0 x_1, x_1 x_2, x_2 x_0 \rangle \), \( A_3 = \langle x_0 x_1 x_2 \rangle \), and \( A_i = 0 \) for all \( i \geq 4 \).

Moreover, \( A_2 \) is contained in the center of \( A \). In particular, \( x_0 x_1 x_2 = x_1 x_2 x_0 = x_2 x_0 x_1 \) in \( A \).
(2) $A$ is an 8-dimensional symmetric $k$-algebra.

(3) $\dim_k(I_j) = \dim_k(J_j) = 4$ for all $j \in \mathbb{Z}$.

(4) $\Omega_1(I_j) = I_{j+1}$ and $\Omega_1(J_{j+1}) = J_j$ for all $j \in \mathbb{Z}$.

(5) The $A$-modules $I_j$ (respectively, $A^{op}$-modules $J_j$) are pairwise non-isomorphic for all $j \in \mathbb{Z}$.

In the next lemma, we calculate dimensions of homomorphism groups related to the modules $I_i$ and $J_i$.

**Lemma 6.5.** Let $i$ and $j$ be integers. Then

1. $I_j$ has a basis $\{x_2 + q^i x_1, x_2 x_0 - q^{-1} x_0 x_1, x_1 x_2, x_0 x_1 x_2\}$, and $J_j$ has a basis $\{x_2 + q^i x_1, x_2 x_0 - q^{-1} x_0 x_1, x_1 x_2, x_0 x_1 x_2\}$.

2. $L(u_j) = I_{j+1}$, $R(u_{j+1}) = J_j$.

3. $J_j \simeq \text{Hom}_A(I_j, A)$.

4. As $k$-vector spaces, $\text{Hom}_A(I_j, I_i) \simeq J_j \cap I_i = \begin{cases} < x_2 + q^i x_1, x_1 x_2, x_0 x_1 x_2 > & \text{if } j = i, \\ < x_0 x_1 x_2 > & \text{if } j = i - 2, \\ < x_1 x_2, x_0 x_1 x_2 > & \text{otherwise}. \end{cases}$

In particular, $\dim_k \text{Hom}_A(I_j, I_i) = \begin{cases} 3 & \text{if } j = i \text{ or } i - 2, \\ 2 & \text{otherwise}. \end{cases}$

5. $\dim_k \text{Ext}_A^1(I_j, I_i) = \begin{cases} 1 & \text{if } j \leq i \leq j + 3, \\ 0 & \text{otherwise}. \end{cases}$

6. $A(1, u_i) = A$ and $A(u_i, 1) = J_i$.

7. $A(u_j, u_i) = \begin{cases} < x_1, x_2, x_0 x_1 x_2 > & \text{if } j = i, \\ < x_2 x_0 - q^{-1} x_0 x_1, x_1 x_2, x_0 x_1 x_2 > & \text{if } j = i - 2, \\ < x_0 x_1 x_2 > & \text{otherwise}. \end{cases}$

**Proof.** (1) and (2). By definition, $I_j = Au_j$. One can check directly that

$$
x_0 u_j = (-q)(x_2 x_0 - q^{-1} x_0 x_1), \quad x_2 u_j = -q^{i+1} x_1 x_2, \quad x_1 u_j = x_1 x_2, \quad x_1 x_2 u_j = x_0 x_1 x_2 u_j = 0, \quad x_0 x_1 u_j = x_0 x_1 x_2, \quad x_2 x_0 u_k = q^i x_0 x_1 x_2.
$$

This implies that $I_j = < x_2 + q^i x_1, x_2 x_0 - q^{-1} x_0 x_1, x_1 x_2, x_0 x_1 x_2 >$. Note that $0 \to L(u_j) \to A \to Au_j \to 0$ is an exact sequence of $A$-modules. Since $u_j u_{j+1} = (x_2 + q^i x_1)(x_2 + q^i x_1) = 0$, we have $I_{j+1} \subseteq L(u_j)$. In addition, $\dim_k I_{j+1} = \dim_k L(u_j) = 4$. It follows that $L(u_j) = I_{j+1}$. Similarly, we can prove the corresponding statements in (1) and (2) for $J_j$.

3. It follows from (2) that $R(L(u_j)) = R(A u_{j+1}) = R(u_{j+1}) = J_j$. By Lemma 6.3(1), we get an isomorphism $\varphi_{u_i} : \text{Hom}_A(I_j, A) \simeq J_j$ of $k$-vector spaces. In fact, we can check directly that $\varphi_{u_i}$ is an isomorphism of $A^{op}$-modules. This proves (3).

4. Note that $\text{Hom}_A(I_j, I_i) = \text{Hom}_A(A u_j, A u_i) \simeq u_j A \cap u_i = J_j \cap I_i$. To prove (4), there are three cases to be considered.

Case 1: $j = i$. By (1) and (2), we conclude that $< x_2 + q^i x_1, x_1 x_2, x_0 x_1 x_2 > \subseteq I_j \cap J_j$. Since $\dim_k(I_j) = 4$ and $x_2 x_0 - q^{i+1} x_0 x_1 \notin I_j$, we get $\dim_k(I_j \cap J_j) = 3$. As a result, $I_j \cap J_j = < x_2 + q^i x_1, x_1 x_2, x_0 x_1 x_2 >$.

Case 2: $j = i - 2$. Note that $x_2 x_0 - q^{i+1} x_0 x_1 = x_2 x_0 - q^{-1} x_0 x_1$. But $x_2 + q^i x_1 \notin I_i$. It follows that $I_i \cap J_i = < x_2 x_0 - q^{i+1} x_0 x_1, x_1 x_2, x_0 x_1 x_2 >$.

Case 3: $j \notin \{i, i - 2\}$. We claim that $I_i \cap J_i = < x_1 x_2, x_0 x_1 x_2 >$. Obviously, $< x_1 x_2, x_0 x_1 x_2 >$ is contained in $I_i \cap J_i$. Conversely, if $\lambda \in I_i \cap J_i$, then there are elements $a_1, a_2, a_3, b_1, b_2, b_3$ such that $\lambda = a_1(x_2 + q^i x_1) + a_2(x_2 x_0 - q^{-1} x_0 x_1) + a_3 x_1 x_2 + a_4 x_0 x_1 x_2 = b_1(x_2 + q^i x_1) + b_2(x_2 x_0 - q^{-1} x_0 x_1) + b_3 x_1 x_2 + b_4 x_0 x_1 x_2$. This implies that $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4 = 0, q^i = q^{-1},$ and $a_4 q^{i+1} = b_4 q^{-1}$. Consequently, $a_1 = a_2 = 0$, which means that $\lambda \in < x_1 x_2, x_0 x_1 x_2 >$. Thus $I_i \cap J_i = < x_1 x_2, x_0 x_1 x_2 >$.

5. The exact sequence $0 \to I_{j+1} \to A \to I_j \to 0$ of $A$-modules induces the following exact sequence of $k$-modules:

$$0 \to \text{Hom}_A(I_j, I_i) \to \text{Hom}_A(A, I_i) \to \text{Hom}_A(I_{j+1}, I_i) \to \text{Ext}_A^1(I_j, I_i) \to 0.$$
By (4), we have
\[ \dim_k \text{Hom}_A(I_j, I_i) = \begin{cases} 
3 & \text{if } i \in \{j, j+2\}, \\
2 & \text{otherwise}.
\end{cases} \]

Since \( \dim_k (I_i) = 4 \), we have
\[ \dim_k \text{Ext}^1_A(I_j, I_i) = \begin{cases} 
1 & \text{if } j \leq i \leq j+3, \\
0 & \text{otherwise}.
\end{cases} \]

This proves (5).

(6) By definition, we know that \( A(1, u_i) = A \), and \( A(u_i, 1) = R(u_{i+1}) = J_i \).

(7) It follows from (4) and Lemma 6.3(2) that
\[ \dim_k A(u_j, u_i) = \begin{cases} 
7 & \text{if } j \in \{i - 2, i\}, \\
6 & \text{otherwise}.
\end{cases} \]

By definition, we know that \( A(u_j, u_i) = \{a \in A \mid u_{j+1} au_i = 0\} \). It is not hard to see that
\[ < x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 > \subseteq A(u_j, u_i). \]

Hence, if \( j \not\in \{i - 2, i\} \), then \( A(u_j, u_i) = < x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 > \). If \( j = i \), then \( u_{j+1} u_j = 0 \), and therefore \( 1 \in A(u_j, u_i) \). Thus \( A(u_j, u_j) = < 1, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 > \). If \( j = i - 2 \), then we can check that \( u_{j+1} x_0 u_{j+2} = 0 \). Thus, \( x_0 \in A_{j,j+2} \). This shows that \( A(u_j, u_j + 2) = < x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 > \).

For higher cohomological groups, we have the following estimation.

**Lemma 6.6.** Let \( t \) be an integer and \( j \) a positive integer Then

1. \( \dim_k \text{Ext}^j_A(I_0, I_i) = \begin{cases} 
1 & \text{if } -1 \leq t - j \leq 2, \\
0 & \text{otherwise}.
\end{cases} \)

2. \( \dim_k \text{Ext}^j_A(I_i, I_0) = \begin{cases} 
1 & \text{if } -2 \leq t + j \leq 1, \\
0 & \text{otherwise}.
\end{cases} \)

3. \( \text{Ext}^j_A(I_0, I_0) = 0 \) for \( j > 1 \).

**Proof.** By Lemma 6.4, we have \( \text{Ext}^j_A(I_0, I_i) \simeq \text{Ext}^j_A(I_0, \Omega^j_A(I_i)) \simeq \text{Ext}^j_A(I_0, I_{j+1}) \). Now (1) follows from Lemma 6.5(5). Similarly, we can prove (2). Clearly, (3) follows from (1) and (2).

Here and subsequently, \( \delta_j \) stands for the canonical exact sequence \( 0 \to I_{j+1} \to A \to I_j \to 0 \) in \( A \)-mod for each \( j \in \mathbb{Z} \).

**Lemma 6.7.** Let \( l \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Then
\[ \{ j \in \mathbb{Z} \mid \delta_j \text{ is an add}(A \oplus I_l)\text{-split sequence in } A\text{-mod} \} = \{ j \in \mathbb{Z} \mid j > l + 2 \text{ or } j < l - 3 \}. \]

In particular, we have
\[ \{ j \in \mathbb{Z} \mid \delta_j \text{ is an add}(A \oplus \bigoplus_{i=0}^n I_i)\text{-split sequence in } A\text{-mod} \} = \{ j \in \mathbb{Z} \mid j > n + 2 \text{ or } j < -3 \}. \]

**Proof.** For any \( j \in \mathbb{Z} \), we know that \( \delta_j \) is an add\((A \oplus I_l)\)-split sequence in \( A\)-mod if and only if \( \text{Ext}^1_A(I_j, I_{j+1}) = \text{Ext}^1_A(I_l, I_j) = 0 \), which is equivalent to the condition that \( j + 1 \not\in [l, l + 3] \) and \( j \not\in [l - 3, l] \) by Lemma 6.3(5). Thus we have (1). Clearly, (2) follows from (1) immediately.

The following result can be directly deduced from the work of Hu and Xi in [6, 7].
Lemma 6.8. Let $B$ be a $k$-algebra. Let $Y$ and $M$ be $B$-modules with $M$ a generator for $B$-mod. If
\[ \text{Ext}^1_B(M, \Omega_B(Y)) = \text{Ext}^1_B(Y, M) = 0, \]
then the endomorphism algebras $\text{End}_B(M \oplus Y)$ and $\text{End}_B(M \oplus \Omega_B(Y))$ are derived equivalent. If, in addition, $\text{Ext}^2_B(M, \Omega_B(Y)) = \text{Ext}^2_B(Y, M) = 0$, then the $\{0,1\}$-Auslander-Yoneda algebras $E^B_{\{0,1\}}(M \oplus Y)$ and $E^B_{\{0,1\}}(M \oplus \Omega_B(Y))$ are derived equivalent.

Having made the preparations, now we can prove Theorem 6.1

Proof of Theorem 6.1 Let $m \geq n + 4$. Set $M := A \oplus I$ with $I = \bigoplus_{i=0}^n I_i$, and $V_m := M \oplus I_m$.

(1) By Lemma 6.5(5), we know that $\text{Ext}^1_A(M, I_m) = \text{Ext}^1_A(I_m, M) = 0$. Clearly, we have
\[ \dim_k (\Lambda_{m}^{(0)}) = \dim_k \text{End}_A(M) + \dim_k \text{Hom}_A(M, I_m) + \dim_k \text{Hom}_A(I_m, M) + \dim_k \text{End}_A(I_m) \]
and
\[ \dim_k (\Lambda_{m}^{(0,1)}) = \dim_k (\Lambda_{m}^{(0)}) + \dim_k \text{Ext}^1_A(M, I_m) + \dim_k \text{Ext}^1_A(I_m, M). \]
By Lemma 6.5, we get
\[ \dim_k \text{End}_A(I_m) = 3, \quad \dim_k \text{Ext}^1_A(I_m, I_m) = 1, \quad \dim_k \text{Hom}_A(M, I_m) = \dim_k \text{Hom}_A(I_m, M) = 2n + 6. \]
It follows that $\dim_k (\Lambda_{m}^{(0,1)}) = 2.\Phi_{m+1}$.

(2) We first show that $\text{gl.dim}(\Lambda_{m}^{(0)}) = \infty$. By Lemma 6.5(5), we have $\text{Ext}^1_A(V_m, I_j) = 0$ for any $j < 0$. Note that, for any $t < j < 0$, there is a long exact sequence
\[ 0 \to I_j \to A \to A \to \cdots \to A \to I_t \to 0. \]
It follows that the induced sequence
\[ 0 \to \text{Hom}_A(V_m, I_j) \to \text{Hom}_A(V_m, A) \to \cdots \to \text{Hom}_A(V_m, A) \to \text{Hom}_A(V_m, I_t) \to 0 \]
is exact. Since $\text{Hom}_A(V_m, A)$ is a projective-injective indecomposable $\Lambda_{m}^{(0)}$-module, we have $\text{inj.dim} \text{Hom}_A(V_m, I_j) = \infty$ for all $j < 0$. Hence $\text{gl.dim}(\Lambda_{m}^{(0)}) = \infty$. Note that there is a canonical surjective homomorphism $\pi : \Lambda_{m}^{(0,1)} \to \Lambda_{m}^{(0)}$ of algebras. Thus every $\Lambda_{m}^{(0,1)}$-module can be regarded as a $\Lambda_{m}^{(0)}$-module. In addition, $E^A_{\{0,1\}}(V_m, A) = \text{Hom}_A(V_m, A)$. It follows that $\text{inj.dim} \text{Hom}_A(V_m, I_j) = \infty$ for all $j < 0$. This yields $\text{gl.dim}(\Lambda_{m}^{(0,1)}) = \infty$.

(3) Recall a classical result on dominant dimension: Let $B$ an algebra and $Y$ a generator-cogenerator for $B$-mod. Suppose that $s$ is a non-negative integer. Then $\text{dom.dim}(\text{End}_B(Y)) = s + 2$ if and only if $\text{Ext}^i_B(Y, Y) = 0$ for all $i$ with $1 \leq i \leq s$, but $\text{Ext}^{i+1}_B(Y, Y) \neq 0$. In our case, we take $Y := V_m$ and $s = 0$. By Lemma 6.5(5), we know that $\text{Ext}^1_A(I_0, I_0) \neq 0$, which means that $\text{Ext}^1_A(V_m, V_m) \neq 0$. Note that $V_m$ is a generator-cogenerator for $A$-mod. Thus $\text{dom.dim}(\Lambda_{m}^{(0)}) = 2$. By Lemma 5.1 we have $\text{dom.dim}(\Lambda_{m}^{(0,1)}) = 0$.

(4) Consider the exact sequence
\[ \delta_m : 0 \to I_{m+1} \to A \to I_m \to 0 \]
in $A$-mod. Since $m \geq n + 4$, it follows from Lemma 6.5(5) and Lemma 6.4(4) that $\text{Ext}^1_A(M, I_{m+1}) = \text{Ext}^1_A(I_{m+1}, M) = \text{Ext}^1_A(I_m, M) = \text{Ext}^1_A(M, I_m) = 0$. Note that $A$ is self-injective. By Lemma 6.8, we conclude that the algebras $\Lambda_{m}^{(0)}$ and $\Lambda_{m+1}^{(0)}$ are derived-equivalent for $\Phi = \{0\}$ or $\{0, 1\}$.

(5) It follows from Lemma 6.4 that $\Omega_A(I_j) = I_{j+1}$ for each $j \in \mathbb{Z}$ and that the $A$-modules $I_j$ are pairwise non-isomorphic for all $j \in \mathbb{Z}$. Now, we define $W := I_0$ and $W_n := \oplus_{0 \leq j \leq n} I_j$. Then, by Corollary 5.3, the algebras $\Lambda_{m}^{(0)}$ and $\Lambda_{m+1}^{(0)}$ are not stably equivalent of Morita type if $j > m$. Thus the proof is completed. \[ \square \]

In the rest of this section, we consider the special case: $n = 0$ and $\Phi = 0$ in Theorem 6.1. For convenience, we set $A_m := \text{End}_A(A \oplus I_0 \oplus I_m)$ for $m \in \mathbb{Z}$, and define $C := \langle x_1, x_2, x_0, x_1, x_2, x_0 \oplus x_0, x_0 \otimes x_0 \rangle$, $T := \langle x_1, x_2, x_0, x_1, x_2, x_0 \rangle$, and $S := T \otimes x_0$. Note that they all are subspaces of $A$. 23
Proposition 6.9. Let $m$ be an integer. Then

1. If $m \neq 2$, then $\Lambda_m$ is isomorphic to the algebra

$$M_A(1, u_0, u_m) := \begin{pmatrix} A & A/I_1 & A/I_{m+1} \\ J_0 & C/I_1 & T/I_{m+1} \\ J_m & T/I_1 & C/I_{m+1} \end{pmatrix}.$$ 

2. $\Lambda_2$ is isomorphic to the algebra

$$M_A(1, u_0, u_2) := \begin{pmatrix} A & A/I_1 & A/I_3 \\ J_0 & C/I_1 & S/I_3 \\ J_2 & T/I_1 & C/I_3 \end{pmatrix}.$$ 

3. Suppose $m \geq 3$. Then, for any $l > m$, the algebras $\Lambda_l$ and $\Lambda_m$ are derived-equivalent, but not stably equivalent of Morita type.

**Proof.** (1) and (2) follow easily from Lemma 6.3 and Lemma 6.5 while (3) can be concluded from Lemma 6.7, Lemma 6.8 and Corollary 5.3. 

For each positive integer $m \geq 3$, the algebra $\Lambda_m$ is given by the following quiver $Q$ with relations $\rho_m$:

$$Q: \quad \begin{array}{c}
\bullet & \bullet & \bullet \\
\gamma_0 & \alpha & \beta \\
3 & 8 & 1 \\
\end{array}$$

$$\rho_m: \quad \alpha^2 = \gamma_0 \beta \gamma_0 \alpha \beta = \gamma_m \alpha \delta \gamma_m \delta = 0;$$

$$\left( \begin{array}{c} \alpha \beta \\
\alpha \delta \gamma_m 
\end{array} \right) \gamma_0 = \frac{1}{1 - q^m} \left( \begin{array}{cc}
q^{m+2} - 1 & 1 - q^2 \\
q^{m+2} - q^m & q^m - q^2
\end{array} \right) \left( \begin{array}{c} \beta \gamma_0 \\
\delta \gamma_m \alpha
\end{array} \right);$$

$$\frac{\beta \gamma_0 \beta}{1 - q} = \frac{\delta \gamma_0 \beta}{q^2 - q^m}, \quad \frac{\beta \gamma_0 \delta}{q^2 - q^m} = \frac{\delta \gamma_0 \delta}{q^m - q^2}.$$ 

The Cartan matrix of $\Lambda_m$ for $m \geq 3$ is

$$C = \begin{pmatrix} 8 & 4 & 4 \\
4 & 3 & 2 \\
4 & 2 & 3 \end{pmatrix},$$

which is symmetric. Moreover, there is an anti-automorphism on $\Lambda_m$ for $(m \geq 3)$, which is given by

$$e_1 \mapsto e_1, \ e_2 \mapsto e_3, \ e_3 \mapsto e_2, \ \beta \mapsto \gamma_m, \ \gamma_m \mapsto q^m \beta, \ \alpha \mapsto \alpha, \ \delta \mapsto \gamma_0, \ \gamma_0 \mapsto \delta.$$ 

It follows from Proposition 6.9 that $\Lambda_t$, $t \geq 3$, are pairwise derived-equivalent, but not stably equivalent of Morita type.

Note that the Cartan matrix of $\Lambda_2$ is not symmetric. Thus $\Lambda_2$ is not derived-equivalent to $\Lambda_m$ for $m \geq 3$ since the Cartan matrices of two derived equivalent algebras are congruent over $\mathbb{Z}$, and therefore derived equivalences preserve the symmetry of Cartan matrices. We don’t know whether $\Lambda_1$ and $\Lambda_3$ are derived-equivalent or not.

It would be interesting to show that the family of algebras in Theorem 6.1 or in Proposition 6.9 are pairwise not stably equivalent.
References


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