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REVIEW ARTICLE

Operator ordering in quantum optics theory and the development of Dirac's symbolic method

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Abstract
We present a general unified approach for arranging quantum operators of optical fields into ordered products (normal ordering, antinormal ordering, Weyl ordering (or symmetric ordering)) by fashioning Dirac's symbolic method and representation theory. We propose the technique of integration within an ordered product (IWOP) of operators to realize our goal. The IWOP makes Dirac's representation theory and the symbolic method more transparent and consequently more easily understood. The beauty of Dirac's symbolic method is further revealed. Various applications of the IWOP technique, such as in developing the entangled state representation theory, nonlinear coherent state theory, Wigner function theory, etc, are presented.

Keywords: Dirac's symbolic method, IWOP technique, entangled state of continuum variables, nonlinear coherent states

1. Introduction

In general, a full description of the states of the photon field in the quantum theory of light is given by the density operators, which are composed of a photon creation operator $a^\dagger$ and an annihilation operator a, with the commutative relation $[a, a^\dagger] = 1$. A number of possible representations for the density operators of the electromagnetic field have been invented [1]. One may convert the density operator in terms of Fock states, or the overcompleteness relation of the coherent state, as c-number functions via the P-function [2, 3], the Wigner function [4] and the Q-function [5]. In all these converting processes, one has to arrange operators in some definite ordering. For example, when the density operator is in normal (antinormal) ordering, its Q-function (P-function) is immediately obtained. Operator ordering is also widely encountered in obtaining miscellaneous optical field states and calculating the expectation values of the operators in these states; for example, one needs normally ordered squeezing operators to construct squeezed states [6]. To our knowledge, in most of the literature there exist two main approaches for handling operator ordering problems, one is the Lie algebra method [7, 8] and the other is Louisell's differential operation method [9] via the coherent state representation [2, 10].

In this review we try to recommend a general unified approach for arranging quantum operators of optical fields into ordered products (normal ordering, antinormal ordering, Weyl ordering (or symmetric ordering)); this is the technique of integration within an ordered product (IWOP) of operators [11–14]. This technique is proposed by fashioning Dirac's symbolic method [15] and representation theory.

The terminology 'symbolic method' was first used in the preface of Dirac's book *The Principles of Quantum Mechanics*: 'The symbolic method, which deals directly in an abstract way with the quantities of fundamental importance…, however, seems to go more deeply into the nature of things. It enables one to express the physical law in a neat and concise way, and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed'.

In this review we start our discussion by posing the following three questions:

(1) How can we better understand Dirac's symbolic method?
(2) How can we develop Dirac's symbolic method, especially his representation theory?
(3) Can we find more applications of Dirac's representation theory, especially in tackling operator ordering problems in quantum optics theory?

In this review we shall show that the IWOP technique can not only help us to better understand the symbolic method, but can also directly develop its special mathematics. As a result, many operator identities and some new quantum mechanical representations can be derived, of which the entangled state representation of continuum variables is the most important. (The conception of quantum entanglement originated with Einstein, Podolsky and Rosen in their argument that quantum mechanics is incomplete. Nowadays it is the essential point of quantum communication.)

This review is arranged as follows: in section 2 we propound problems which we did not fully understand about Dirac's symbolic method. In section 3 we introduce the IWOP technique for fundamental Bose operators and show the concrete process of performing the integral in equation (2). In section 4 we recast the completeness relation of some quantum mechanical representations into normally ordered Gaussian forms by virtue of the IWOP technique, which further reveals the beauty of Dirac's symbolic method. In section 5, based on the normally ordered coordinate (momentum) eigenstate projector $|q\rangle\langle q|$ ($|p\rangle\langle p|$), we introduce the Wigner operator $\Delta(p,q)$ in a natural way. Using the IWOP technique we obtain the coherent state representation and the explicit form of $\Delta(p,q)$. Sections 6 and 7 are devoted to deriving operator ordering formulae and studying the operator Fredholm equation by virtue of the IWOP technique. In section 8, using the same technique, we introduce the coordinate-momentum intermediate representation $|q,\mu,\nu\rangle$ and point out its relation to the Radon transform of the Wigner operator. In section 9, we derive the generalized normally ordered n-mode squeezing operator. In sections 10 and 11 we introduce the technique of integration within the antinormally ordered (Weyl ordered) product of Bose operators and derive the antinormally ordered (Weyl ordered) expansion of density matrices in a coherent state basis, respectively. In section 12, using the IWOP technique, we derive some unitary operators as the direct mapping of classical canonical transforms. In section 13 using the IWOP technique we introduce the entangled state representation of continuum variables in which the two-mode squeezing operator has a natural representation. In section 14 we show that the IWOP technique can also be extended to the nonlinear coherent state case.

2. Propounding problems

Let us begin by looking at Dirac's representation theory, in which quantum variables have to be abstracted from their matrix representation and turned into the purely symbolic notion of q-numbers. In particular, the coordinate eigenstates' completeness relation has a simpler form

$$\int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle\langle q| = 1, \tag{1}$$

where $|q\rangle$ is an eigenstate of the coordinate operator Q, $Q|q\rangle = q|q\rangle$; equation (1) looks symmetric with respect to ket and bra. Nevertheless, we feel that our understanding of the representation theory is very shallow because we do not know the 'value' of the slightly changed integral

$$S_1 \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\mu}} \left|\frac{q}{\mu}\right\rangle\langle q|, \qquad \mu > 0, \tag{2}$$

where the ket and bra are not symmetric. We deem the integral meaningful, as $|q/\mu\rangle$ is in the set of eigenstates of the coordinate operator. But we do not know how to perform this integral in a neat and direct way. So two questions arise: why not execute it directly since it is an integral and what physical meaning does S_1 have?

Generalizing the problem involved in (1) we are also confronted with the question of how to directly perform the following ket–bra form integral

$$\begin{gathered} U_2 \equiv \int\int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{matrix} q_1 \\ q_2 \end{matrix}\right|, \\ \left\langle\begin{matrix} q_1 \\ q_2 \end{matrix}\right| \equiv \langle q_1, q_2|, \qquad AD - BC = 1, \end{gathered} \tag{3}$$

in a concise way. These two integrations have their own physical meaning. The first one manifestly shows the mapping of the classical scaling transformation $q \to q/\mu$ into a quantum mechanical unitary squeezing operator, while the second one indicates how the classical canonical transformation $(q_1,q_2) \to (Aq_1 + Bq_2, Cq_1 + Dq_2)$ maps into its corresponding unitary operator. This operator may be used in discussing quantum gate operation. At this point we mention that in Dirac's book the analogy between classical and quantum mechanics was not limited to Heisenberg's formal transposition of the Newtonian equations. For Dirac the analogy involved deeper-lying structural properties, those classically expressed in the algebra of Poisson brackets. As for the quantum equation of motion, the beautiful link with classical mechanics through the Poisson brackets and the correspondence between classical canonical transform and quantum unitary transform were undisputedly Dirac's very own contributions, which established him on the international scene. Here we further express this correspondence through nonsymmetric ket–bra operator integral, once the integral is explicitly performed, the unitary operator is obtained. Extending (2) and (3) to the multimode case, we consider the $2n \times 2n$ symplectic transformation $\boldsymbol{G}$ keeping the classical Poisson bracket invariant. The well-known symplectic condition is

$$\boldsymbol{G}\begin{pmatrix} 0 & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & 0 \end{pmatrix}\tilde{\boldsymbol{G}} = \tilde{\boldsymbol{G}}\begin{pmatrix} 0 & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & 0 \end{pmatrix}\boldsymbol{G} = \begin{pmatrix} 0 & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & 0 \end{pmatrix}, \tag{4}$$

where $\boldsymbol{I}_n$ is an $n \times n$ unit matrix and $\boldsymbol{G}$ is a $2n \times 2n$ matrix,

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}, \tag{5}$$

which causes a symplectic transformation in $2n$-dimensional ($\vec{q} - \vec{p}$) phase space

$$\begin{pmatrix} \vec{q}\,' \\ \vec{p}\,' \end{pmatrix} = \boldsymbol{G}\begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}. \tag{6}$$

(For more information and references about symplectic groups we refer the reader to [56].) In terms of (5) the symplectic property involved in equation (4) is expressed as

$$\boldsymbol{A}\tilde{\boldsymbol{D}} - \boldsymbol{B}\tilde{\boldsymbol{C}} = \boldsymbol{I}_n, \qquad \boldsymbol{A}\tilde{\boldsymbol{B}} = \boldsymbol{B}\tilde{\boldsymbol{A}}, \qquad \boldsymbol{C}\tilde{\boldsymbol{D}} = \boldsymbol{D}\tilde{\boldsymbol{C}}, \tag{7}$$

$$\tilde{\boldsymbol{A}}\boldsymbol{D} - \tilde{\boldsymbol{C}}\boldsymbol{B} = \boldsymbol{I}_n, \qquad \tilde{\boldsymbol{B}}\boldsymbol{D} = \tilde{\boldsymbol{D}}\boldsymbol{B}, \qquad \tilde{\boldsymbol{A}}\boldsymbol{C} = \tilde{\boldsymbol{C}}\boldsymbol{A}. \tag{8}$$

The symplectic transformations form a symplectic group. In this work for establishing a one-to-one correspondence between symplectic transformation and its quantum mechanical image via the coherent state representation, we change the discussion in $(\vec{q} - \vec{p})$ phase space to $(\vec{z} - \vec{z}^*)$ complex space, here $\vec{z} = (\vec{q} + \mathrm{i}\vec{p})/\sqrt{2}$, such that equation (6) is converted to

$$\begin{pmatrix} \vec{z}' \\ \vec{z}'^* \end{pmatrix} = \boldsymbol{K} \begin{pmatrix} \vec{z} \\ \vec{z}^* \end{pmatrix}, \tag{9}$$

where

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{S} & -\boldsymbol{R} \\ -\boldsymbol{R}^* & \boldsymbol{S}^* \end{pmatrix}. \tag{10}$$

The relationship between $\boldsymbol{G}$ and $\boldsymbol{K}$ is

$$\begin{aligned} \boldsymbol{A} &= \frac{\boldsymbol{S} + \boldsymbol{S}^* - \boldsymbol{R} - \boldsymbol{R}^*}{2}, \qquad \boldsymbol{B} = \frac{\boldsymbol{S}^* + \boldsymbol{R}^* - \boldsymbol{S} - \boldsymbol{R}}{2\mathrm{i}}, \\ \boldsymbol{C} &= \frac{\boldsymbol{S} - \boldsymbol{S}^* - \boldsymbol{R} + \boldsymbol{R}^*}{2\mathrm{i}}, \qquad \boldsymbol{D} = \frac{\boldsymbol{S} + \boldsymbol{R} + \boldsymbol{S}^* + \boldsymbol{R}^*}{2}. \end{aligned} \tag{11}$$

Accordingly, equations (7) and (8) are equivalent to

$$\boldsymbol{S}\boldsymbol{S}^\dagger - \boldsymbol{R}\boldsymbol{R}^\dagger = \boldsymbol{I}_n, \qquad \boldsymbol{R}\tilde{\boldsymbol{S}} = \boldsymbol{S}\tilde{\boldsymbol{R}}, \tag{12}$$

$$\boldsymbol{S}^\dagger\boldsymbol{S} - \tilde{\boldsymbol{R}}\boldsymbol{R}^* = \boldsymbol{I}_n, \qquad \boldsymbol{R}^\dagger\boldsymbol{S} = \tilde{\boldsymbol{S}}\boldsymbol{R}^*, \tag{13}$$

while the symplectic condition (4) is equivalent to

$$\boldsymbol{K}^\dagger \begin{pmatrix} \boldsymbol{I}_n & 0 \\ 0 & -\boldsymbol{I}_n \end{pmatrix} \boldsymbol{K} = \boldsymbol{K} \begin{pmatrix} \boldsymbol{I}_n & 0 \\ 0 & -\boldsymbol{I}_n \end{pmatrix} \boldsymbol{K}^\dagger = \begin{pmatrix} \boldsymbol{I}_n & 0 \\ 0 & -\boldsymbol{I}_n \end{pmatrix}. \tag{14}$$

Then what is the operator image of the classical symplectic transformation $\vec{z} \to \boldsymbol{S}\vec{z} - \boldsymbol{R}\vec{z}^*$ in phase space by a phase-space integral over coherent states

$$\begin{aligned} U(\boldsymbol{S}, \boldsymbol{R}) &\equiv \sqrt{|\det(\boldsymbol{S})|} \int \frac{\mathrm{d}^2\vec{z}}{\pi^n} |\boldsymbol{S}\vec{z} - \boldsymbol{R}\vec{z}^*\rangle\langle\vec{z}| \\ &= \sqrt{|\det \boldsymbol{S}|} \int \frac{\mathrm{d}^2\vec{z}}{\pi^n} \left| \begin{pmatrix} \boldsymbol{S} & -\boldsymbol{R} \\ -\boldsymbol{R}^* & \boldsymbol{S}^* \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{z}^* \end{pmatrix} \right\rangle \left\langle \begin{matrix} \vec{z} \\ \vec{z}^* \end{matrix} \right| = ? \end{aligned} \tag{15}$$

where

$$|\vec{z}\rangle = \prod_i \exp[-|z_i|^2/2 + z_i a_i^\dagger]|0\rangle_i, \tag{16}$$

is the n-mode coherent state? All these problems challenge us to find an effective method to directly perform c-number integration over the ket–bra projection operators. In the next section we shall demonstrate how the IWOP technique can fulfil this task and lead to explicitly normally ordered unitary operators.

3. The IWOP technique

To deal with these integrations, let us introduce the IWOP technique. We begin by listing some properties of normal products of operators which means all the creation operators stand on the left of the annihilation operators in a monomial of $a^\dagger$ and a:

(1) The order of Bose operators a and $a^\dagger$ within a normally ordered product can be permuted. That is to say, even though $[a, a^\dagger] = 1$, we can have $:aa^\dagger: = :a^\dagger a: = a^\dagger a$, where $:\,:$ denotes normal ordering.

(2) c numbers can be taken out of the symbol $:\,:$ as one wishes.

(3) The symbol $:\,:$ which is within another symbol $:\,:$ can be deleted.

(4) A normally ordered product can be integrated or differentiated with respect to a c-number provided the integration is convergent.

(5) The vacuum projection operator $|0\rangle\langle 0|$ has the normal product form

$$|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}:, \tag{17}$$

where $a|0\rangle = 0$ (in Louisell's book [9], $|0\rangle\langle 0| = \lim_{\epsilon=1} :\mathrm{e}^{-\epsilon a^\dagger a}:$). A rigorous proof for (17) is as follows. Suppose $|0\rangle\langle 0| = :W:$, W is to be determined. From the completeness relation of the Fock state

$$\begin{aligned} &\sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \qquad |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle, \\ &N|n\rangle = n|n\rangle, \qquad N = a^\dagger a, \end{aligned} \tag{18}$$

we have

$$\begin{aligned} 1 &= \sum_{n,n'=0}^{\infty} |n\rangle\langle n'| \frac{1}{\sqrt{n!n'!}} \left(\frac{\mathrm{d}}{\mathrm{d}Z^*}\right)^n (Z^*)^{n'}|_{Z^*=0} \\ &= \mathrm{e}^{a^\dagger \frac{\mathrm{d}}{\mathrm{d}Z^*}}|0\rangle\langle 0|\mathrm{e}^{Z^* a}|_{Z^*=0} = :\mathrm{e}^{a^\dagger a}W: = :\mathrm{e}^{a^\dagger a}:W::, \end{aligned} \tag{19}$$

which, together with property 3 above, gives equation (17). From (17) and property 1 we have

$$\begin{aligned} |0\rangle\langle 0| &= :\sum_{n=0}^{\infty} \frac{(-a^\dagger a)^n}{n!}: = \sum_{n=0}^{\infty} \frac{(-)^n a^{\dagger n} a^n}{n!} \\ &= 1 - N + \frac{1}{2!}N(N-1) \\ &\quad - \frac{1}{3!}N(N-1)(N-2) + \cdots. \end{aligned} \tag{20}$$

(6) The Hermitian operation can 'run across' the 'border' of $:\,:$ and can directly apply to bosonic operators as the following

$$:(W \cdots V):^\dagger = :(W \cdots V)^\dagger:, \tag{21}$$

where $W, \ldots, V$ are Bose operators.

(7) The differentiation with respect to a or $a^\dagger$ within normal ordering symbols possesses the property

$$\begin{aligned} &:\frac{\partial}{\partial a} f(a, a^\dagger): = [:f(a, a^\dagger):, a^\dagger], \\ &:\frac{\partial}{\partial a^\dagger} f(a, a^\dagger): = [a, :f(a, a^\dagger):]. \end{aligned} \tag{22}$$

Now we turn to directly performing the integral (2). Readers can see shortly that using the IWOP technique its result is just the normally ordered single-mode squeezing operator. Because the Fock representation of $|q\rangle$ is [15],

$$\langle n|q\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \mathrm{e}^{-\frac{m\omega}{2\hbar}q^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right), \tag{23}$$

where H_n is the Hermitian polynomial with its generating function being

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n = \mathrm{e}^{-t^2+2xt}, \tag{24}$$

we have

$$|q\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|q\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \times \exp\left\{-\frac{m\omega}{2\hbar}q^2 + \sqrt{\frac{2m\omega}{\hbar}} q a^\dagger - \frac{a^{\dagger 2}}{2}\right\}|0\rangle. \tag{25}$$

(In the following, unless particularly mentioned, we take $\hbar = \omega = m = 1$ for convenience.) Using (25) we can write (2) as

$$S_1 \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\mu}} \left|\frac{q}{\mu}\right\rangle\langle q| = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi\mu}} \mathrm{e}^{-\frac{q^2}{2\mu^2} + \sqrt{2}\frac{q}{\mu}a^\dagger - \frac{a^{\dagger 2}}{2}} |0\rangle\langle 0| \mathrm{e}^{-\frac{q^2}{2} + \sqrt{2}qa - \frac{a^2}{2}}. \tag{26}$$

Substituting (17) into (26) we see

$$S_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi\mu}} \mathrm{e}^{-\frac{q^2}{2\mu^2} + \sqrt{2}\frac{q}{\mu}a^\dagger - \frac{a^{\dagger 2}}{2}} {:}\mathrm{e}^{-a^\dagger a}{:} \mathrm{e}^{-\frac{q^2}{2} + \sqrt{2}qa - \frac{a^2}{2}}. \tag{27}$$

Note that on the left of $:\mathrm{e}^{-a^+ a}:$ are all creation operators, while on its right are all annihilation operators, so the whole integral is in normal ordering; thus using property 1 we have

$$S_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi\mu}} {:}\mathrm{e}^{-\frac{q^2}{2}(1+\frac{1}{\mu^2}) + \sqrt{2}q(\frac{a^\dagger}{\mu}+a) - \frac{1}{2}(a+a^\dagger)^2}{:}. \tag{28}$$

As a commutes with $a^\dagger$ within : :, so $a^\dagger$ and a can be considered as if they were parameters while the integration is performing. Therefore, by setting $\mu = \mathrm{e}^\lambda$, $\operatorname{sech}\lambda = \frac{2\mu}{1+\mu^2}$, $\tanh\lambda = \frac{\mu^2+1}{\mu^2-1}$, we are able to perform the integration and obtain

$$S_1 = \sqrt{\frac{2\mu}{1+\mu^2}} {:}\exp\left\{\frac{(\frac{a^\dagger}{\mu}+a)^2}{1+\frac{1}{\mu^2}} - \frac{1}{2}(a+a^\dagger)^2\right\}{:} = (\operatorname{sech}\lambda)^{1/2} \mathrm{e}^{-\frac{a^{\dagger 2}}{2}\tanh\lambda} {:}\mathrm{e}^{(\operatorname{sech}\lambda - 1)a^\dagger a}{:} \mathrm{e}^{\frac{a^2}{2}\tanh\lambda}, \tag{29}$$

which is just the single-mode squeezing operator in normal ordering appearing in many references [6, 7]. It is worth mentioning that we have not used the $SU(1, 1)$ Lie algebra method in the derivation. The integration automatically arranges the squeezing operator in normal ordering. Using

$$\mathrm{e}^{\lambda a^\dagger a} = \sum_{n=0}^{\infty} \mathrm{e}^{\lambda n}|n\rangle\langle n| = \sum_{n=0}^{\infty} \mathrm{e}^{\lambda n}\frac{a^{\dagger n}}{n!} {:}\mathrm{e}^{-a^\dagger a}{:} a^n = {:}\exp[(\mathrm{e}^\lambda - 1)a^\dagger a]{:}, \tag{30}$$

equation (29) becomes

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\mu}} \left|\frac{q}{\mu}\right\rangle\langle q| = \mathrm{e}^{-\frac{a^{+2}}{2}\tanh\lambda} \mathrm{e}^{(a^+ a+\frac{1}{2})\ln\operatorname{sech}\lambda} \mathrm{e}^{\frac{a^2}{2}\tanh\lambda}. \tag{31}$$

So far we have just used $\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\mu}} |\frac{q}{\mu}\rangle\langle q|$ as an example to show how the IWOP works. It inspires a physical interpretation of some of the mathematical quantities employed in the theory: the classical dilation $q \to \frac{q}{\mu}$ manifestly maps into the normally ordered squeezing operator. It also shows that the fundamental representation theory can be formulated in a not so abstract way, as we can now directly perform the integral over ket–bra projection operators. Moreover, the IWOP technique can be employed to perform many complicated integrations for ket–bra projection operators.

There is a large gap between quantum mechanical operators theory (q-numbers) and classical numbers theory(c-numbers). The IWOP technique arranges noncommutable operators within an ordered product symbol in a way that they become commutable, in this sense the 'gap' between q-numbers and c-numbers is 'narrowed'. However, the nature of the operators is not changed, they are still q-numbers, not c-numbers. After the integration over c-numbers within an ordered product is performed, we can finally get rid of the normal ordering symbol by using (30).

4. Completeness relation of some representations re-obtained by virtue of the IWOP technique

In particular, for $\mu = 1$, equation (28) becomes

$$\int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} {:}\mathrm{e}^{-q^2+2q(\frac{a+a^\dagger}{\sqrt{2}}) - \frac{1}{2}(a+a^\dagger)^2}{:} = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} {:}\mathrm{e}^{-(q-Q)^2}{:} = 1 \tag{32}$$

(a real simple Gaussian integration!) where $Q = \frac{a+a^\dagger}{\sqrt{2}}$. This immediately leads us to put the completeness relation of the momentum representation into the normally ordered Gaussian form

$$\int_{-\infty}^{\infty} \mathrm{d}p\, |p\rangle\langle p| = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{\pi}} {:}\mathrm{e}^{-(p-P)^2}{:} = 1, \tag{33}$$

where $P = \frac{a-a^\dagger}{\sqrt{2}\mathrm{i}}$, and $|p\rangle$ is the momentum eigenvector

$$|p\rangle = \pi^{-\frac{1}{4}} \exp[-\tfrac{1}{2}p^2 + \mathrm{i}\sqrt{2}pa^\dagger + \tfrac{1}{2}a^{\dagger 2}]|0\rangle. \tag{34}$$

From the above calculation we can better understand Dirac's symbolic method and further develop it. Consequently, there are many applications of the IWOP technique. For example, the coherent state's overcompleteness relation can be rewritten as

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| = \int \frac{\mathrm{d}^2 z}{\pi} {:}\mathrm{e}^{-(z^*-a^\dagger)(z-a)}{:} = 1, \quad \mathrm{d}^2 z \equiv \mathrm{d}(\operatorname{Re} z)\mathrm{d}(\operatorname{Im} z), \tag{35}$$

where at the final step we have use the mathematical formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\{\lambda|z|^2 + fz + gz^*\} = -\frac{1}{\lambda}\exp\left(-\frac{fg}{\lambda}\right), \quad \operatorname{Re}\lambda < 0. \tag{36}$$

The displaced Fock state is constructed by operating the displacing operator $D(z) = \exp(za^\dagger - z^* a)$ on the Fock state $|n\rangle$,

$$|z, n\rangle = D(z)|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger - z^*)^n|z\rangle. \tag{37}$$

Its completeness can easily be seen by virtue of the IWOP too [16],

$$\int \frac{d^2z}{\pi}|z,m\rangle\langle z,n| = \int \frac{d^2z}{\pi}{:}\frac{1}{\sqrt{m!n!}}(a^\dagger - z^*)^m(a-z)^n e^{-(z^*-a^\dagger)(z-a)}{:} = \frac{1}{\sqrt{m!n!}}\int \frac{d^2z}{\pi} z^{*m}z^n e^{-|z|^2}(-1)^{m+n} = \delta_{m,n}I. \quad (38)$$

Using the IWOP we can find some new optical field state with the completeness relation, for instance the so-called displacement and squeezing related state [17]

$$|z\rangle_g = \exp\left[-\frac{|z|^2}{2} + (fz \pm z^*g)a^\dagger \mp fga^{\dagger 2}\right]|0\rangle, \quad (39)$$

where $|f|^2 + |g|^2 = 1$, is complete, as

$$\int \frac{d^2z}{\pi}|z\rangle_{gg}\langle z| = \int \frac{d^2z}{\pi}{:}\exp[-|z|^2 + z(fa^\dagger \pm g^*a) + z^*(f^*a \pm ga^\dagger) \mp fga^{\dagger 2} \mp f^*g^*a^2 - a^\dagger a]{:} = {:}\exp[(|f|^2+|g|^2-1)a^\dagger a]{:} = 1. \quad (40)$$

We would say at this point that what we have done here is to improve or extend the mathematical formulation of representation theory, with which new complete quantum states having physical meaning can be found. Also, we would like to say that the pursuit of mathematical beauty in quantum optics theory is our 'guiding light' in this research.

5. Wigner operator: explicit form, coherent state representation, studied by virtue of the IWOP technique

Enlightened by (32) and (33) we can compose the following operator

$$\frac{1}{\pi}{:}e^{-(q-X)^2-(p-P)^2}{:} = \Delta(q,p), \quad (41)$$

which satisfies

$$\int\int_{-\infty}^{\infty} dq\, dp\, \Delta(q,p) = 1. \quad (42)$$

Its marginal integration is

$$\int_{-\infty}^{\infty} dp\, \Delta(q,p) = \frac{1}{\sqrt{\pi}}{:}e^{-(q-Q)^2}{:} = |q\rangle\langle q|,$$
$$\int_{-\infty}^{\infty} dq\, \Delta(q,p) = \frac{1}{\sqrt{\pi}}{:}e^{-(p-P)^2}{:} = |p\rangle\langle p|, \quad (43)$$

respectively. According to Wigner's original idea of setting up a function in $q-p$ phase space whose marginal distributions are the probability of finding a particle in coordinate space and momentum space respectively, we can immediately judge that the operator $\Delta(q,p)$ in (41) is just the Wigner operator. We can also prove that the coherent state representation of $\Delta(q,p)$ is [18]

$$\Delta(q,p) \to \Delta(\alpha,\alpha^*) = \int \frac{d^2z}{\pi^2}|\alpha+z\rangle\langle\alpha-z|, e^{\alpha z^* - z\alpha^*}, \quad \alpha = \frac{1}{\sqrt{2}}(q+ip), \quad (44)$$

where $|z\rangle$ is the coherent state, the integral measure $d^2z \equiv d(\mathrm{Re}\,z)\,d(\mathrm{Im}\,z)$ as in (35). In fact, using the IWOP technique to perform the integral of (41) yields its explicit form

$$\Delta(\alpha,\alpha^*) = \int \frac{d^2z}{\pi^2}{:}\exp\{-|z|^2 + (\alpha+z)a^\dagger + (\alpha^*-z^*)a - \alpha^*a - a^\dagger a + \alpha z^* - z\alpha^*\}{:} = \frac{1}{\pi}{:}e^{-2(\alpha^*-a^\dagger)(\alpha-a)}{:} = \frac{1}{\pi}e^{-2|\alpha|^2-2a^\dagger\alpha}{:}e^{-2a^\dagger a}{:}\, e^{-2\alpha^* a}. \quad (45)$$

Because

$$\int_{-\infty}^{\infty} dq\,|-q\rangle\langle q| = \int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}{:}e^{-q^2+\sqrt{2}q(-a^+ + a)-\frac{1}{2}(a+a^+)^2}{:} = {:}e^{-2a^+a}{:} = e^{i\pi N} \quad (46)$$

is a parity operator, (45) becomes

$$\Delta(\alpha,\alpha^*) = \frac{1}{\pi}e^{-2|\alpha|^2-2a^\dagger\alpha}(-)^N e^{-2\alpha^* a} = \frac{1}{\pi}D(2\alpha)(-)^N. \quad (47)$$

Equation (45) or (47) is the explicit form of the Wigner operator. Recently it was reported that the Wigner function defined via the parity operator can be directly measured [19]. Some new experiments on the reconstruction of Wigner functions have been performed by Mlynek's group [20]. Using (44) the Weyl quantization scheme [21], which takes the Wigner operator as an integral kernel to transit a classical function $h(p,q) \equiv f(\alpha^*,\alpha)$, $\alpha = \frac{1}{\sqrt{2}}(q+ip)$, to operator $H(P,Q) \equiv F(a^\dagger,a)$,

$$H(P,Q) = \int\int_{-\infty}^{\infty} dp\, dq\, h(p,q)\Delta(p,q), \quad (48)$$

can be converted to

$$F(a^\dagger,a) = 2\int d^2\alpha\, f(\alpha^*,\alpha)\Delta(\alpha,\alpha^*). \quad (49)$$

The Wigner theorem states that expectation values $\langle F\rangle$ of certain operators F can be expressed as integrals similar to the phase-space integrals of classical probability theory,

$$\langle F\rangle \equiv \mathrm{Tr}[\rho F(a^\dagger,a)] = 2\int d^2\alpha\, \mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)]f(\alpha^*,\alpha), \quad (50)$$

where $\mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)]$ is known as the Wigner function of the density operator ρ. The differential element $2d^2\alpha$ (the measure of the integration) in (50) is a real element of an area proportional to the phase-space element $dq\,dp$ in classical mechanics; this point has been emphasized in [22] by Bužek *et al*. Especially, when $\rho = |z\rangle\langle z|$, the pure coherent state, its Wigner function is

$$\langle z|\Delta(\alpha,\alpha^*)|z\rangle = \frac{1}{\pi}\langle z|{:}e^{-2(\alpha^*-a^\dagger)(\alpha-a)}{:}|z\rangle = \frac{1}{\pi}e^{-2(\alpha^*-z^*)(\alpha-z)}. \quad (51)$$

Up to a factor of 2π equation (51) is the same as equation (2.16a) in [22], so our definition of the Wigner operator differs from that of [22] by a factor of 2π. The converse relation of (49), using (44), is

$$f(\alpha^*,\alpha) = 2\pi\,\mathrm{Tr}[F\Delta(\alpha,\alpha^*)] = 2\int\frac{d^2z}{\pi} \times \langle\alpha-z|F|\alpha+z\rangle e^{\alpha z^* - z\alpha^*}, \quad (52)$$

from which one can obtain the classical Weyl function of the operator F.

In section 8 we shall derive the Radon transform of the Wigner operator $\Delta(\alpha,\alpha^*)$.

6. Operator ordering formulae derived by the IWOP technique

Now we show that the IWOP technique and Dirac's representation can help us to derive many operator ordering formulae in a unified and concise way. Using the overcompleteness relation (32) of the coherent state and the mathematical formula

$$\int \frac{d^2z}{\pi} z^n z^{*m} e^{A|z|^2+Bz+Cz^*} = e^{-BC/A} \times \sum_{l=0} \frac{n!m!}{l!(n-l)!(m-l)!(-A)^{n+m-l+1}} B^{m-l}C^{n-l}, \quad \mathrm{Re}\,A<0 \tag{53}$$

as well as the IWOP we can directly put operator $a^m a^{\dagger n}$ into antinormal ordering,

$$a^n a^{\dagger m} = \int \frac{d^2z}{\pi} z^n z^{*m} |z\rangle\langle z| = \int \frac{d^2z}{\pi} z^n z^{*m} {:}e^{-|z|^2+za^\dagger+z^*a-a^\dagger a}{:} = \sum_{l=0}^{\min(m,n)} {:}\frac{n!m!}{l!(n-l)!(m-l)!} a^{\dagger m-l}a^{n-l}{:}. \tag{54}$$

Furthermore, using the mathematical formula

$$\int \frac{d^2z}{\pi} \exp(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}) = \frac{1}{\sqrt{\zeta^2-4fg}} \exp\left[\frac{-\zeta\xi\eta+\xi^2 g+\eta^2 f}{\zeta^2-4fg}\right], \tag{55}$$

whose convergent condition is either

$$\mathrm{Re}(\zeta+f+g)<0, \qquad \mathrm{Re}\left(\frac{\zeta^2-4fg}{\zeta+f+g}\right)<0, \tag{56}$$

or

$$\mathrm{Re}(\zeta-f-g)<0, \qquad \mathrm{Re}\left(\frac{\zeta^2-4fg}{\zeta-f-g}\right)<0, \tag{57}$$

we can rearrange the following antinormal products into normal ordering

$$e^{fa^2}e^{ga^{\dagger 2}} = \int \frac{d^2z}{\pi} e^{fz^2}|z\rangle\langle z| e^{gz^{*2}} = \int \frac{d^2z}{\pi} {:}\exp(-|z|^2+za^\dagger+z^*a+fz^2+gz^{*2}-a^\dagger a){:} = \frac{1}{\sqrt{1-4fg}} \exp[ga^{\dagger 2}/(1-4fg)] \times \exp[-a^\dagger a \ln(1-4fg)] \exp[fa^2/(1-4fg)], \tag{58}$$

and

$$a^n e^{\nu a^{\dagger 2}} = \int \frac{d^2z}{\pi} z^n |z\rangle\langle z| e^{\nu z^{*2}} = \int \frac{d^2z}{\pi} {:}\exp(-|z|^2+za^\dagger+z^*a+\nu z^{*2}-a^\dagger a)z^n{:} = e^{\nu a^{\dagger 2}} \sum_{k=0}^{[n/2]} \frac{n!\nu^k}{k!(n-2k)!} {:}(2\nu a^\dagger+a)^{n-2k}{:}. \tag{59}$$

Using the IWOP we can also derive some operator-disentangling identities. For example, due to (32) we have

$$\begin{aligned}\exp(\lambda a^\dagger a+\sigma a^2) &= \int \frac{d^2z}{\pi} \exp(\lambda a^\dagger a+\sigma a^2) e^{za^\dagger} \times \exp(-\lambda a^\dagger a-\sigma a^2)|0\rangle\langle z| e^{-|z|^2/2} \\ &= \int \frac{d^2z}{\pi} \exp\left\{-|z|^2/2+z\left(a^\dagger e^\lambda + a\frac{2\sigma}{\lambda}\sinh\lambda\right)\right\}|0\rangle\langle z| \\ &= \int \frac{d^2z}{\pi} {:}\exp\left\{-|z|^2+z^*a+za^\dagger e^\lambda + \frac{\sigma e^\lambda}{\lambda}z^2\sinh\lambda - a^\dagger a\right\}{:} \\ &= e^{\lambda a^\dagger a}\exp\left\{a^2\frac{\sigma e^\lambda}{\lambda}\sinh\lambda\right\}.\end{aligned} \tag{60}$$

This method can be extended to the multimode case too, using the mathematical formula [23]

$$\begin{aligned}&\int \prod_{i=1}^n \left[\frac{d^2z_i}{\pi}\right] \exp\left\{-\frac{1}{2}(z \quad z^*) \begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} z \\ z^*\end{pmatrix} + (\mu \quad \nu^*)\begin{pmatrix} z \\ z^*\end{pmatrix}\right\} \\ &= \left[\det\begin{pmatrix} C & D \\ A & B\end{pmatrix}\right]^{-\frac{1}{2}} \times \exp\left[\tfrac{1}{2}(\mu \quad \nu^*)\begin{pmatrix} C & D \\ A & B\end{pmatrix}^{-1}\begin{pmatrix} \nu^* \\ \mu\end{pmatrix}\right],\end{aligned} \tag{61}$$

where $(z \quad z^*) = (z_1, z_2, \ldots, z_n \quad z_1^*, z_2^*, \ldots, z_n^*)$, and using the IWOP technique we can do normal ordering for the following multimode operators:

$$\begin{aligned}\exp[a_i\sigma_{ij}a_j]\exp[a_i^\dagger\tau_{ij}a_j^\dagger] &= \int \prod_{i=1}^n \left[\frac{d^2z_i}{\pi}\right] \exp(a_i\sigma_{ij}a_j) \times |z_1,z_2,\ldots,z_n\rangle\langle z_1,z_2,\ldots,z_n| \exp(a_i^\dagger\tau_{ij}a_j^\dagger) \\ &= \int \prod_{i=1}^n \left[\frac{d^2z_i}{\pi}\right] {:}\exp[-z_i^*z_i+a_i^\dagger z_i + a_jz_j^*+z_i\sigma_{ij}z_j+z_i^*\tau_{ij}z_j^*-a_i^\dagger a_i]{:} \\ &= \int \prod_{i=1}^n \left[\frac{d^2z_i}{\pi}\right] {:}\exp\left[-\frac{1}{2}(z,z^*)\begin{pmatrix} -2\sigma & I_n \\ I_n & -2\tau\end{pmatrix} \times \begin{pmatrix} z \\ z^*\end{pmatrix} + (a^\dagger a)\begin{pmatrix} z \\ z^*\end{pmatrix} - a_i^\dagger a_i\right]{:} \\ &= \left[\det\begin{pmatrix} I_n & -2\tau \\ -2\sigma & I_n\end{pmatrix}\right]^{-\frac{1}{2}} \times {:}\exp\left[\tfrac{1}{2}(a^\dagger \quad a)\begin{pmatrix} I_n & -2\tau \\ -2\sigma & I_n\end{pmatrix}^{-1}\begin{pmatrix} a \\ a^\dagger\end{pmatrix} - a^\dagger a\right]{:} \\ &= [\det(I_n-4\sigma\tau)]^{-\frac{1}{2}}\exp\{a_i^\dagger[(I_n-4\sigma\tau)^{-1}\tau]_{ij}a_j^\dagger\} \\ &\quad \times {:}\exp[a_i^\dagger(I_n-4\sigma\tau)_{ij}^{-1}a_j^\dagger - a_i^\dagger a_i{:} \\ &\quad \times \exp\{a_i[(I_n-4\sigma\tau)^{-1}\sigma]_{ij}a_j\}.\end{aligned} \tag{62}$$

In particular, for the two-mode case we have Q.1

$$\exp[\mu a_1a_2]\exp[\nu a_1^\dagger a_2^\dagger] = \frac{1}{1-\mu\nu}\exp\left(\frac{\nu a_1^\dagger a_2^\dagger}{1-\mu\nu}\right) \times \exp[-(a_1^\dagger a_1+a_2^\dagger a_2)\ln(1-\mu\nu)]\exp\left(\frac{\mu a_1a_2}{1-\mu\nu}\right). \tag{63}$$

Now we want the normally ordered form of $\exp[a_i^\dagger \Lambda_{ij} a_j]$, where Λ is an $n \times n$ matrix. Using the coherent state's completeness relation and the IWOP technique we have

$$\exp(a_i^\dagger \Lambda_{ij} a_j) = \int \prod_{i=1}^{n} \left[\frac{\mathrm{d}^2 z_i}{\pi}\right] \times \exp(a_i^\dagger \Lambda_{ij} a_j) \exp(a_i^\dagger z_i) \exp(-a_i^\dagger \Lambda_{ij} a_j) \times |0, 0 \ldots, 0\rangle\langle z_1, z_2, \ldots, z_n| \exp(-|z_i|^2/2)$$
$$= \int \prod_{i=1}^{n} \left[\frac{\mathrm{d}^2 z_i}{\pi}\right] :\exp[-|z_i|^2 + a_i^\dagger (\mathrm{e}^\Lambda)_{ij} z_j + z_i^* a_i - a_i^\dagger a_i]:$$
$$= :\exp\{a_i^\dagger [\mathrm{e}^\Lambda - I]_{ij} a_j\}:. \qquad (64)$$

As its application we can derive the explicit form of the class operator of the $SU(2)$ rotation group [24].

7. Normally ordered Fredholm operator equation

By considering $\mathrm{e}^{-(x-y)^2}$ as an integral kernel of the Fredholm equation [25]

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-(x-y)^2} \varphi(y) = f(x), \qquad (65)$$

we can construct the corresponding normally ordered operator Fredholm integral equation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q :\exp[-(q-Q)^2]: \varphi(q) = :f(q):. \qquad (66)$$

On the other hand, using (30) we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q :\exp[-(q-Q)^2]: \varphi(q) = \int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle\langle q| \varphi(Q) = \varphi(Q), \qquad (67)$$

so we know the normally ordered expansion of $\varphi(Q)$ is

$$\varphi(Q) = :f(q):. \qquad (68)$$

In this way we can obtain some new operator identities. For example, from (66) we have the normally ordered expansion of Q^n,

$$Q^n = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} q^n :\mathrm{e}^{-(q-Q)^2}: = \frac{1}{\sqrt{\pi}} \sum_{l=0}^{[n/2]} \binom{n}{2l} :\left(\frac{a+a^\dagger}{\sqrt{2}}\right)^{n-2l}: \times \Gamma(l+\tfrac{1}{2}),\ \Gamma(r+\tfrac{1}{2}) = \sqrt{\pi} 2^{-r}(2r-1), \qquad (69)$$

where we have used the mathematical formulae from the handbook [26]

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} \mathrm{e}^{-\sigma(q-\lambda)^2} q^n = \frac{1}{\sqrt{\sigma^{n+1}}} \times \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k} k!(n-2k)!} (\sigma^{\frac{1}{2}} \lambda)^{n-2k}, \qquad \mathrm{Re}\,\sigma > 0. \qquad (70)$$

For another example, from the integral formula about the single variable Hermitian polynomial

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2} H_n(x)\, \mathrm{d}x = \sqrt{\pi}(2y)^n, \qquad (71)$$

we know the normally ordered expansion of $H_n(Q)$,

$$H_n(Q) = \int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle\langle q| H_n(q) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} : \exp[-(q-Q)^2] H_n(q) = 2^n :Q^n:. \qquad (72)$$

Comparing (72) with $H_n'(q) = 2nH_{n-1}(q)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}Q} :Q^n: = 2^{-n} \frac{\mathrm{d}}{\mathrm{d}Q} H_n(Q) = n 2^{1-n} H_{n-1}(Q) = n :Q^{n-1}:, \qquad (73)$$

which means $\frac{\mathrm{d}}{\mathrm{d}Q} :Q^n: = :\frac{\mathrm{d}}{\mathrm{d}Q} Q^n:$. This is another property of normal ordering, which will be useful in deriving the normally ordered expansion of some coordinate operator functions. We now show that solution to the Fredholm equation (66) is

$$\varphi(Q) = :f(Q): = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{2^n n!} H_n(Q). \qquad (74)$$

In fact, substituting the following expansions

$$:\mathrm{e}^{-(q-Q)^2}: = \mathrm{e}^{-q^2} \sum_{n=0}^{\infty} :H_n(q) \frac{Q^n}{n!}:, \qquad \varphi(q) = \sum_{m=0}^{\infty} b_m H_m(q), \qquad (75)$$

into (66) we have

$$\pi^{-1/2} \sum_{n,m=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}q :\mathrm{e}^{-q^2} H_n(q) H_m(q) \frac{Q^n}{n!} b_m: = \sum_{m=0}^{\infty} 2^m b_m :Q^m: = :f(Q):. \qquad (76)$$

Taking the coherent state expectation value for (76), we see

$$\sum_{m=0}^{\infty} 2^m b_m \langle z| :Q^m: |z\rangle = \langle z| :f(Q): |z\rangle. \qquad (77)$$

After differentiating both sides m times with respect to $\mathrm{Re}\, z$ and then setting $\mathrm{Re}\, z = 0$, we obtain $f^{(m)}(0) = m! \delta_{n,m} b_m$, thus

$$\varphi(q) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{2^n n!} H_n(q), \qquad (78)$$

and (74) is proved. Using (32) and the IWOP technique we can also derive

$$\mathrm{e}^{\lambda Q^2} = \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{\lambda q^2} |q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} :\mathrm{e}^{-(q-Q)^2 + \lambda q^2}: = \frac{1}{\sqrt{1-\lambda}} : \exp\left[\frac{\lambda}{1-\lambda} Q^2\right]:, \qquad (79)$$

where $\mathrm{Re}\, \lambda < 1$. Operating both sides of (79) on the vacuum state we immediately know

$$\mathrm{e}^{\lambda Q^2} |0\rangle = \frac{1}{\sqrt{1-\lambda}} \exp\left[\frac{\lambda}{2(1-\lambda)} a^{\dagger 2}\right] |0\rangle, \qquad (80)$$

which belongs to a squeezed state. This implies that when an extra potential λQ^2 is added to a harmonic oscillator potential, a squeezing effect related to a frequency jump will be generated. Similarly, we can use (33) to derive

$$\mathrm{e}^{\lambda P^2} = \frac{1}{\sqrt{1-\lambda}} {:}\exp\left[\frac{\lambda}{1-\lambda}P^2\right]{:}. \tag{81}$$

Letting $z = x + \mathrm{i}y$, using the IWOP technique we can perform the one-sided integral for coherent states,

$$\int_{-\infty}^{\infty} \mathrm{d}x\, |z\rangle\langle z| = \sqrt{\pi}\exp\left(\frac{a^{\dagger 2}}{4} + \mathrm{i}ya^\dagger\right) \times \exp\left(a^\dagger a \ln\frac{1}{2}\right)\exp\left(\frac{a^2}{4} - \mathrm{i}ya - y^2\right) = \sqrt{2\pi}\exp[-(P-\sqrt{2}y)^2], \tag{82}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}y\, |z\rangle\langle z| = \sqrt{2\pi}\exp[-(Q-\sqrt{2}x)^2], \tag{83}$$

which can be used in analysing the properties of a superposition of coherent states on a line. Extending to the two-mode case we have

$$\mathrm{e}^{\lambda Q_1 Q_2} = \frac{1}{\sqrt{1-\lambda^2}} {:}\exp\left[\frac{4}{1-\lambda^2}(Q_1^2+Q_2^2+\lambda Q_1 Q_2)\right]{:}, \tag{84}$$

which, once acting upon the two-mode vacuum state, will yield a two-mode squeezed state.

8. Generalized squeezing operator in normal ordering

We now demonstrate how to convert the generalized squeezing operator $U(G)$ into normal ordering,

$$U(G) = \exp\{\tfrac{1}{2}(a_i G_{ij} a_j - a_i^\dagger G_{ij}^\dagger a_j^\dagger)\}, \quad i, j = 1, 2, \ldots, n, \tag{85}$$

where the repeated indices mean the summation from 1 to n. G is a complex symmetric matrix. Using the Baker–Hausdorff formula we see that $U(G)$ causes the following transform

$$U a_i^\dagger U^{-1} = a_j^\dagger[\cosh(G^\dagger G)^{\frac{1}{2}}]_{ji} + a_j[G(G^\dagger G)^{-\frac{1}{2}}\sinh(G^\dagger G)^{\frac{1}{2}}]_{ji} = a_j^\dagger[\cosh(G^\dagger G)^{\frac{1}{2}}]_{ji} + a_j[\sinh(G^\dagger G)^{\frac{1}{2}}(G^\dagger G)^{-\frac{1}{2}}G]_{ji}. \tag{86}$$

From matrix theory we know that any square matrix can be subject to polar decomposition

$$G = H\mathrm{e}^{\mathrm{i}F}, \tag{87}$$

where both H and F are Hermitian matrices. Due to $G = \tilde{G}$,

$$G = \mathrm{e}^{\mathrm{i}\tilde{F}}\tilde{H}, \qquad \widetilde{G^\dagger G} = GG^\dagger, \qquad \widetilde{GG^\dagger G} = GG^\dagger G, \qquad GG^\dagger = H^2, \qquad G^\dagger G = \tilde{H}^2, \qquad \tilde{H}^2\mathrm{e}^{-\mathrm{i}F} = \mathrm{e}^{\mathrm{i}F}H^2, \tag{88}$$

so we can re-form (86) as

$$U a_i^\dagger U^{-1} = a_j^\dagger(\cosh\tilde{H})_{ji} + a_j[(\mathrm{e}^{\mathrm{i}\tilde{F}}\sinh\tilde{H})]_{ji} = a_j^\dagger(\cosh\tilde{H})_{ji} + a_j[(\sinh H)\,\mathrm{e}^{\mathrm{i}F}]_{ji}. \tag{89}$$

Similarly we derive

$$U a_i U^{-1} = a_j(\cosh H)_{ji} + a_j^\dagger[(\mathrm{e}^{-\mathrm{i}F}\sinh H)]_{ji}. \tag{90}$$

Let $\|0\rangle = U|0\rangle$ be the new vacuum state, using $U^\dagger(G) = U(-G)$ we have

$$\begin{aligned} a_i\|0\rangle &= UU^{-1}a_i U|0\rangle \\ &= U[a_j(\cosh H)_{ji} - a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\sinh H)_{ji}]|0\rangle \\ &= -\,U a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\sinh H)_{ji}U^{-1}U|0\rangle \\ &= -\,[a_l^\dagger(\cosh\tilde{H}\mathrm{e}^{-\mathrm{i}F}\sinh H)_{li} + a_l(\sinh^2 H)_{li}]\|0\rangle. \end{aligned} \tag{91}$$

It then follows from

$$a_i\|0\rangle = -a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ji}\|0\rangle, \tag{92}$$

that this is the equation the new vacuum state should obey. Due to

$$[a_i, \exp[-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger]] = -\,a_j^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ji}\exp[-\tfrac{1}{2}a_l^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{lj}a_j^\dagger], \tag{93}$$

so the solution to (92) is

$$\|0\rangle = c\exp[-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger]|0\rangle \tag{94}$$

where c is the normalization constant, determined by $1 = \langle 0\|0\rangle$, using the IWOP technique and (62) we can derive

$$c = [\det(\operatorname{sech} H)]^{\frac{1}{2}}. \tag{95}$$

Now we can study how the n-mode coherent state $|\vec{z}\rangle$ changes under the U transform

$$\begin{aligned} U|\vec{z}\rangle &= U\exp(z_i a_i^\dagger)U^{-1}U|0\rangle\exp(-\tfrac{1}{2}|z_i|^2) \\ &= [\det(\operatorname{sech} H)]^{\frac{1}{2}}\exp\{z_i[a_j^\dagger(\cosh\tilde{H})_{ji} + a_j(\sinh H\mathrm{e}^{\mathrm{i}F})_{ji}]\} \\ &\quad\times\exp\{-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger - \tfrac{1}{2}|z_i|^2\}|0\rangle \\ &= [\det(\operatorname{sech} H)]^{\frac{1}{2}}\exp(-\tfrac{1}{2}|z_i|^2 + z_i(\operatorname{sech} H)_{ij}a_j^\dagger \\ &\quad + \tfrac{1}{2}z_i(\tanh H\mathrm{e}^{\mathrm{i}F})_{ij}z_j - \tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger]|0\rangle. \end{aligned} \tag{96}$$

Using the completeness relation of the coherent state and the **Q.3** IWOP technique we finally derive the disentangling of $U(G)$,

$$\begin{aligned} U &= \int\frac{\mathrm{d}^2 z}{\pi}U|\vec{z}\rangle\langle\vec{z}| = [\det(\operatorname{sech} H)]^{\frac{1}{2}}\int \mathrm{i}\Pi\left[\frac{\mathrm{d}^2 z}{\pi}\right] \\ &\quad\times{:}\exp\{-|z_i|^2 + z_i(\operatorname{sech} H)a_j^\dagger + z_i^* a_i \\ &\quad + \tfrac{1}{2}z_i(\tanh H\mathrm{e}^{\mathrm{i}F})_{ij}z_j - \tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger - a_i^\dagger a_i\}{:} \\ &= [\det(\operatorname{sech} H)]^{\frac{1}{2}}\int \mathrm{i}\Pi\left[\frac{\mathrm{d}^2 z_i}{\pi}\right] \\ &\quad\times{:}\exp\left\{-\tfrac{1}{2}(z, z^*)\begin{pmatrix}-(\tanh H)\,\mathrm{e}^{\mathrm{i}F} & I \\ I & 0\end{pmatrix}\begin{pmatrix}z \\ z^*\end{pmatrix}\right. \\ &\quad + (a^\dagger\operatorname{sech}\tilde{H}, a)\begin{pmatrix}z \\ z^*\end{pmatrix} - a_i^\dagger a_i \\ &\quad\left. - \tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger\right\}{:} \\ &= [\det(\operatorname{sech} H)]^{\frac{1}{2}}\exp[-\tfrac{1}{2}a_i^\dagger(\mathrm{e}^{-\mathrm{i}F}\tanh H)_{ij}a_j^\dagger] \\ &\quad\times{:}\exp[a_i^\dagger(\operatorname{sech}\tilde{H} - I)_{ij}a_j]{:}\exp[\tfrac{1}{2}a_i(\tanh H\mathrm{e}^{\mathrm{i}F})_{ij}a_j. \end{aligned} \tag{97}$$

Comparing (96) with (29) we see the similarity. Thus $U(G)$ **Q.4** is the n-mode generalized squeezing operator.

9. Coordinate-momentum intermediate representation $|q,\mu,\nu\rangle$ and Radon transform of the Wigner operator

It was Dirac who first introduced the representation theory into quantum mechanics. Different representations fit the solution of different dynamical problems; as Dirac pointed out: 'When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible'. In this section we shall reveal that there exists a new quantum mechanical representation composed of the eigenvector $|q,\mu,\nu\rangle$ of the operator $\mu Q+\nu P$, which is inherent to the Radon transformation of the Wigner operator. In quantum optical theory $\mu Q+\nu P$ (μ and ν are real numbers) represents all possible linear combination of quadratures Q and P of the oscillator field mode a and can be measured by the homodyne measurement just by varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of the Wigner function, thus the homodyne measurement of the light field permits the reconstruction of the Wigner function of a quantum system by varying the phase of the local oscillator. Vogel and Risken [27] and Smithey *et al* [28] pointed out that the probability distribution for the rotated quadrature phase can be expressed in terms of the Wigner function, and the reverse is also true, i.e. one can obtain the Wigner distribution by tomographic inversion of a set of measured marginal probability distributions of the quadrature amplitude. In the following we shall show that the introduction of the explicit form of $|q,\mu,\nu\rangle$ will reduce other abstract quantities concerning the quadrature $\mu Q+\nu P$, such as Radon transformation of the Wigner operator to a form which is as simple as possible. We show that the projector $|q,\mu,\nu\rangle\langle q,\mu,\nu|$ is just the Radon transformation of the Wigner operator with (μ,ν) being Radon transformation parameters; we shall use the normally ordered form of the Wigner operator to demonstrate this relation. For a review of Radon transform, we refer the reader to [29].

The explicit form of $|q,\mu,\nu\rangle$ in Fock space is

$$|q,\mu,\nu\rangle=[\pi(\mu^2+\nu^2)]^{-1/4}\exp\left\{-\frac{q^2}{2(\mu^2+\nu^2)}+\sqrt{2}\frac{q}{\mu-\mathrm{i}\nu}a^\dagger-\frac{\mathrm{i}\nu+\mu}{2(\mu-\mathrm{i}\nu)}a^{\dagger 2}\right\}|0\rangle. \tag{98}$$

It is not difficult to see that $|q,\mu,\nu\rangle$ satisfies the eigenvector equation

$$(\mu Q+\nu P)|q,\mu,\nu\rangle=q|q,\mu,\nu\rangle. \tag{99}$$

Remarkably, as a result of (17) and (98), $|q,\mu,\nu\rangle\langle q,\mu,\nu|$ has a neat normally ordered Gaussian form

$$|q,\mu,\nu\rangle\langle q,\mu,\nu|=\frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\times{:}\exp\left\{-\frac{1}{\mu^2+\nu^2}[q-(\mu Q+\nu P)]^2\right\}{:}. \tag{100}$$

Then with use of the IWOP technique we prove that $|q,\mu,\nu\rangle$ form a complete set,

$$\int_{-\infty}^{\infty}\mathrm{d}q\,|q,\mu,\nu\rangle\langle q,\mu,\nu|=1, \tag{101}$$

and possess the orthonormal property

$$\langle q',\mu,\nu|q'',\mu,\nu\rangle=\delta(q'-q''), \tag{102}$$

so the set of $|q,\mu,\nu\rangle$ makes up a new representation interpolating the coordinate representation and the momentum representation. Using the explicit normally ordered form of Wigner operator (41) we perform the Radon transform

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{d}p'\,\mathrm{d}q'\,\delta(q-\nu p'-\mu q')\Delta(p',q')\\&=\int_{-\infty}^{+\infty}\mathrm{d}p'\,\mathrm{d}q'\,\delta(q-\nu p'-\mu q')\frac{1}{\pi}{:}\mathrm{e}^{-(p'-P)^2-(q'-Q)^2}{:}\\&=\frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}{:}\mathrm{e}^{-\frac{1}{\mu^2+\nu^2}[q-(\mu Q+\nu P)]^2}{:}=|q,\mu,\nu\rangle\langle q,\mu,\nu|,\end{aligned} \tag{103}$$

which is just equation (99). We conclude that the projection operator $|q,\mu,\nu\rangle\langle q,\mu,\nu|$ is just the Radon transformation of the Wigner operator; μ,ν are Radon transformation parameters, so the introduction of $|q,\mu,\nu\rangle$ is necessary. From the point of view of Weyl quantization scheme we see that $\delta(q-\nu p'-\mu q')$ is just the classical Weyl correspondence of the operator $|q,\mu,\nu\rangle\langle q,\mu,\nu|$.

10. The technique of integration within the antinormally ordered product of Bose operators and the antinormally ordered expansion of density matrices in a coherent state basis

When all creation operators $a^\dagger$ stand on the right of a, we say that they are in antinormal ordering. Moreover, the IWOP technique can be extended to the antinormal product case. Introducing the generalized IWOP technique for antinormally ordering Bose operators, the main points are:

(1) The order of Bose operators a and $a^\dagger$ within an antinormally ordered product $\vdots\ \vdots$ are permuted. That is to say, even though $[a,a^\dagger]=1$, we can have

$$aa^\dagger=\vdots aa^\dagger\vdots=\vdots a^\dagger a\vdots. \tag{104}$$

(2) The symbol $\vdots\ \vdots$ which is within another symbol $\vdots\ \vdots$ can be deleted.

(3) An antinormally ordered product can be integrated or differentiated with respect to a c-number provided the integration is convergent.

(4) The antinormal ordering form of the vacuum projection operator is

$$|0\rangle\langle 0|=\pi\delta(a)\delta(a^\dagger)=\int\frac{\mathrm{d}^2\eta}{\pi}\exp[\mathrm{i}\eta a]\exp[\mathrm{i}\eta^* a^\dagger]. \tag{105}$$

Proof.

$$\begin{aligned}\pi\delta(z-a)\delta(z^*-a^\dagger)&=\int\frac{\mathrm{d}^2\eta}{\pi}\times\exp[-\mathrm{i}\eta(z-a)]\exp[-\mathrm{i}\eta^*(z^*-a^\dagger)]\\&=\int\frac{\mathrm{d}^2\eta}{\pi}\,\vdots\exp[-|\eta|^2-\mathrm{i}\eta(z-a)-\mathrm{i}\eta^*(z^*-a^\dagger)]\vdots\\&=\vdots\exp[-(a^\dagger-z^*)(a-z)]\vdots=|z\rangle\langle z|.\end{aligned} \tag{106}$$

□

Q.5 When $z = 0$, (106) reduces to (105). From (105) we can also write the coherent state projector as

$$|z\rangle\langle z| = \int \frac{d^2\eta}{\pi} \exp(za^\dagger)\exp[i\eta a]\exp[i\eta^* a^\dagger]\exp[z^* a - |z|^2]$$
$$= \int \frac{d^2\eta}{\pi} \vdots\exp[i\eta a + i\eta^* a^\dagger + z^* a + za^\dagger]$$
$$\times \exp(-2|z|^2 - iz\eta - iz^*\eta^*)\vdots. \qquad (107)$$

Now we recall the Glauber–Sudarshan P-representation in the coherent state basis

$$\rho = \int \frac{d^2 z}{\pi} P(z)|z\rangle\langle z|. \qquad (108)$$

Its inverse relation is given by Mehta [30]

$$P(z) = e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \exp(|\beta|^2 + \beta^* z - z^*\beta), \qquad (109)$$

where $|\beta\rangle$ is a coherent state too. Now substituting (109) into (108) and using (107) we can expand ρ as

$$\rho = \int \frac{d^2 z}{\pi} e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} \exp(\beta^* z - z^*\beta)|z\rangle\langle z|\vdots$$
$$= \vdots \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \int \frac{d^2\eta}{\pi} \exp[i\eta a + i\eta^* a^\dagger]$$
$$\times \int \frac{d^2 z}{\pi} \exp[-|z|^2 - iz\eta - iz^*\eta^*$$
$$+ \beta^* z - z^*\beta + z^* a + za^\dagger]\vdots$$
$$= \vdots \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + aa^\dagger]\vdots. \qquad (110)$$

Q.6 In particular, when $\rho = 1$, $\langle -\beta||\beta\rangle = \exp[-2|\beta|^2]$,

$$1 = \vdots \int \frac{d^2\beta}{\pi} \exp[-|\beta|^2 + \beta^* a - \beta a^\dagger + aa^\dagger]\vdots. \qquad (111)$$

Readers may compare (111) with (35) and see the difference. As an application of (111), we have

$$e^{-\lambda a^\dagger a} = \vdots \int \frac{d^2\beta}{\pi} \langle -\beta|e^{-\lambda a^\dagger a}|\beta\rangle$$
$$\times \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + aa^\dagger]\vdots$$
$$= \vdots \int \frac{d^2\beta}{\pi} \exp[-e^{\lambda}|\beta|^2 + \beta^* a - \beta a^\dagger + aa^\dagger]\vdots$$
$$= e^{-\lambda}\vdots\exp[(1 - e^{-\lambda})aa^\dagger]\vdots \qquad (112)$$

which provides a new approach for putting an antinormally ordered operator into a normal product.

11. About the technique of integration within a Weyl ordered product of operators

The Weyl ordering accompanies the Weyl quantization rule, which quantizes a classical coordinate-momentum function $q^m p^r$ as

$$q^m p^r \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l. \qquad (113)$$

The right-hand side of (113) is just a Weyl ordered operator. We introduce the symbol $\genfrac{}{}{0pt}{}{:}{:}\ \genfrac{}{}{0pt}{}{:}{:}$ as Weyl ordering, then the Weyl quantization rule in (48) can be recast into

$$\genfrac{}{}{0pt}{}{:}{:} h(P, Q) \genfrac{}{}{0pt}{}{:}{:} = \int\int_{-\infty}^{\infty} dp\, dq\, h(p, q)\Delta(p, q), \qquad (114)$$

which means the classical correspondence of a Weyl ordered operator $\genfrac{}{}{0pt}{}{:}{:} h(P, Q) \genfrac{}{}{0pt}{}{:}{:}$ can be directly obtained by replacing $Q \to q$, $P \to p$. Taking the operator in (113) as an example, since it is in Weyl ordered form, so

$$\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l$$
$$= \genfrac{}{}{0pt}{}{:}{:} \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l \genfrac{}{}{0pt}{}{:}{:}. \qquad (115)$$

Then according to (114) we have

$$\genfrac{}{}{0pt}{}{:}{:} \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l \genfrac{}{}{0pt}{}{:}{:} = \int\int_{-\infty}^{\infty} dp\, dq \left(\frac{1}{2}\right)^m$$
$$\times \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} q^{m-l} p^r q^l \Delta(p, q)$$
$$= \int\int_{-\infty}^{\infty} dp\, dq\, q^m p^r \Delta(p, q), \qquad (116)$$

which complies with (113) and (115). Based on the Weyl quantization rule and the Wigner operator, the technique of integration within a Weyl ordered product (IWWOP) of operators is introduced. The Weyl ordering symbol possesses two remarkable properties:

(1) the order of Bose operators within a Weyl ordered product can be permuted;
(2) a Weyl ordered product can be integrated with respect to a c-number provided that the integration is convergent.

According to (114), the Weyl ordered form of the Wigner operator is,

$$\Delta(p, q) = \genfrac{}{}{0pt}{}{:}{:} \delta(p - P)\delta(q - Q) \genfrac{}{}{0pt}{}{:}{:}. \qquad (117)$$

Since $Q = \frac{a + a^\dagger}{\sqrt{2}}$, $P = \frac{a - a^\dagger}{\sqrt{2}i}$, $\alpha = \frac{q + ip^\dagger}{\sqrt{2}}$, so

$$\Delta(\alpha, \alpha^*) = \frac{1}{2} \genfrac{}{}{0pt}{}{:}{:} \delta(a^\dagger - \alpha^*)\delta(a - \alpha) \genfrac{}{}{0pt}{}{:}{:}. \qquad (118)$$

Let $h(p, q) = f(\alpha, \alpha^*)$, then (114) is equivalent to

$$\genfrac{}{}{0pt}{}{:}{:} f(a, a^\dagger) \genfrac{}{}{0pt}{}{:}{:} = 2 \int d^2\alpha\, f(\alpha, \alpha^*)\Delta(\alpha, \alpha^*). \qquad (119)$$

An interesting question thus naturally arises: what is the Weyl ordering form for an arbitrary given operator ρ? In other words, how do we put ρ into Weyl ordering? We shall present a new formula which can conveniently recast operators into their Weyl ordering. For this purpose, we try to convert the overcompleteness of the coherent state into Weyl ordering. The classical Weyl corresponding function of $|z\rangle\langle z|$ is

$$2\pi\, \mathrm{Tr}[|z\rangle\langle z|\, \Delta(\alpha, \alpha^*)]$$
$$= 2\langle z|{:}\exp[-2(z^* - a^\dagger)(z - a)]{:}|z\rangle$$
$$= 2\exp[-2(z^* - \alpha^*)(z - \alpha)]. \qquad (120)$$

According to (119), the Weyl ordered form of the coherent state projector is

$$|z\rangle\langle z| = 2\int d^2\alpha \exp[-2(z^* - \alpha^*)(z-\alpha)] \times \vdots\,\delta(a^\dagger - \alpha^*)\delta(a-\alpha)\,\vdots = 2\,\vdots\exp[-2(z^* - a^\dagger)(z-a)]\vdots. \quad (121)$$

The completeness relation can also be recast into Weyl ordering,

$$\int \frac{d^2z}{\pi}|z\rangle\langle z| = 2\int\frac{d^2z}{\pi}\,\vdots\exp[-2(z^*-a^\dagger)(z-a)]\vdots = 1. \quad (122)$$

Substituting (121) and (109) into (108) yields

$$\rho = 2\int\frac{d^2z}{\pi}e^{|z|^2}\int\frac{d^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\exp[|\beta|^2 + z\beta^* - \beta z^*] \times \vdots\exp[-2(z^*-a^\dagger)(z-a)]\vdots = 2\int\frac{d^2\beta}{\pi}\,\vdots\langle-\beta|\rho|\beta\rangle\exp[2(\beta^*a - a^\dagger\beta + a^\dagger a)]\vdots. \quad (123)$$

This is the Weyl ordered expansion of operator ρ [31]. Especially when $\rho = 1$, the identity operator, equation (123) reduces to

$$1 = 2\int\frac{d^2\beta}{\pi}\,\vdots\exp[-2(\beta^*+a^\dagger)(\beta-a)]\vdots. \quad (124)$$

It is interesting to compare (124) with the antinormally ordered formula (111). As a first example of applying the formula (123) we can prove

$$2{:}\exp[-2(\beta^*+a^\dagger)(\beta-a)]{:} = \vdots\exp[-|\beta|^2 + \beta^*a - a^\dagger\beta + a^\dagger a]\vdots,$$

where:: denotes normal ordering. For another example, when ρ is a normally ordered operator as

$$\rho = \frac{2}{\pi}{:}\exp[-2(z^*-a^\dagger)(z-a)]{:}, \quad (125)$$

then its Weyl ordered form is

$$\frac{2}{\pi}\int\frac{d^2\beta}{\pi}\,\vdots\langle-\beta|{:}\exp[-2(z^*-a^\dagger)(z-a)]{:} \times|\beta\rangle\exp[2(\beta^*a - a^\dagger\beta + a^\dagger a)]\vdots = \exp[-2z^*z]\frac{2}{\pi}\int\frac{d^2\beta}{\pi} \times\vdots\exp[2\beta(z^*-a^\dagger) - 2\beta^*(z-a) + 2a^\dagger a]\vdots = \tfrac{1}{2}\vdots\,\delta(a^\dagger-\alpha^*)\delta(a-\alpha)\,\vdots, \quad (126)$$

which is just the single-mode Wigner operator. In particular, when $\alpha = 0$, equation (126) reduces to the parity operator's Weyl ordering form

$$(-1)^N = {:}\exp[-2a^\dagger a]{:} = \tfrac{1}{2}\vdots\,\delta(a^\dagger)\delta(a)\,\vdots. \quad (127)$$

In the two-mode case, when $\rho = \exp(fa^\dagger b^\dagger)\exp(gab)$, using (123) we see that its Weyl ordering is

$$\exp(fa^\dagger b^\dagger)\exp(gab) = 4\int\frac{d^2\beta_1\,d^2\beta_2}{\pi^2} \times\exp(-2|\beta_1|^2 - 2|\beta_2|^2 + f\beta_1^*\beta_2^* + g\beta_1\beta_2) \times\vdots\exp[2(\beta_1^*a - a^\dagger\beta_1 + a^\dagger a) + 2(\beta_2^*b - b^\dagger\beta_2 + b^\dagger b)]\vdots = \vdots\frac{2}{4-gf}\exp\left[\frac{1}{4-gf}[4gba + 4fa^\dagger b^\dagger - 2gf(a^\dagger a + b^\dagger b)]\right]\vdots. \quad (128)$$

12. Unitary operators derived by the IWOP technique

Using the IWOP technique and Dirac's representations we can derive many normally ordered unitary operators which are the images of classical transforms. For example, performing the integral

$$\int_{-\infty}^{+\infty}dq\,|q\rangle\langle p||_{p=q} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dq \times{:}\exp\left\{-q^2 + \sqrt{2}(a^\dagger - ia)q + \frac{a^2 - a^{\dagger 2}}{2} - a^\dagger a\right\}{:} = {:}e^{-(i+1)a^\dagger a}{:} = \exp\left(-i\frac{\pi}{2}N\right), \qquad N = a^\dagger a, \quad (129)$$

we obtain the mutual transform $|q\rangle \to |p\rangle$

$$\exp(i\pi N/2)|q\rangle = |p\rangle|_{p=q}, \qquad \exp(i\pi N/2)|p\rangle = |-q\rangle|_{q=p}, \quad (130)$$

and

$$\exp(i\pi N/2)Q\exp(-i\pi N/2) = P, \qquad \exp(i\pi N/2)P\exp(-i\pi N/2) = -Q. \quad (131)$$

Now we turn to the unitary transform which is the quantum image of a two-dimensional classical canonical transform given by (3); using the IWOP we derive

$$U = \frac{2}{\sqrt{L}}\exp\left\{\frac{1}{2L}[(A^2+B^2-C^2-D^2) \times(a_1^{\dagger 2} - a_2^{\dagger 2}) + 4(AC+BD)a_1^\dagger a_2^\dagger]\right\} \times{:}\exp\left\{(a_1^\dagger, a_2^\dagger)(g-I)\begin{pmatrix}a_1\\a_2\end{pmatrix}\right\}{:} \times\exp\left\{\frac{1}{2L}[(B^2+D^2-A^2-C^2) \times(a_1^2 - a_2^2) - 4(AB+DD)a_1a_2]\right\}, \quad (132)$$

where $L = A^2+B^2+C^2+D^2+2$,

$$g = \frac{2}{L}\begin{pmatrix}A+D & B-C\\ B-C & A+D\end{pmatrix}, \qquad \det g = \frac{4}{L}, \qquad I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}. \quad (133)$$

Using (132) we have

$$Ua_1U^{-1} = \tfrac{1}{2}[(A+D)a_1 + (C-B)a_2 - (A-D)a_1^\dagger - (B-C)a_2^\dagger];$$
$$Ua_2U^{-1} = \tfrac{1}{2}[(B-C)a_1 + (A+D)a_2 - (B+C)a_1^\dagger - (A-D)a_2^\dagger]; \quad (134)$$

so

$$UQ_1U^{-1} = DQ_1 - BQ_2,$$
$$UQ_2U^{-1} = -CQ_1 + AQ_2;$$
$$UP_1U^{-1} = AP_1 + CP_2,$$
$$UP_2U^{-1} = BP_1 + DP_2; \quad (135)$$

and

$$U|00\rangle = \frac{2}{\sqrt{L}}\exp\left\{\frac{1}{2L}[(A^2+B^2-C^2-D^2) \times (a_1^{\dagger 2} - a_2^{\dagger 2}) + 4(AC+BD)a_1^\dagger a_2^\dagger]\right\}|00\rangle, \quad (136)$$

which is a generalized two-mode squeezed state. For the unitary transform operators corresponding to the $2n \times 2n$ symplectic transformation $\boldsymbol{G}$ (see equations (4)–(8)) gained by virtue of the IWOP technique, we refer to [32].

From the above discussions we conclude that starting from Dirac's fundamental representation and the IWOP technique we can derive many unitary operators as the direct mapping of classical canonical transforms. The IWOP in some sense is a 'bridge' between classical canonical transformations and quantum mechanical unitary transformations. It can be widely used in transformation theory.

13. Entangled state representation of a continuum variable and two-mode squeezing operator

Since the publication of the paper by Einstein, Podolsky and Rosen (EPR) in 1935 [33], arguing the incompleteness of quantum mechanics, the conception of entanglement has become more and more fascinating and important as it plays a central role in quantum communication and quantum computation [34, 35].

It is known now that by entanglement one means that the two-particle state does not factor into a product of single particle states, but is a sum of at least two terms, each of which is a product. One may recapitulate the entanglement in more mathematical language [36]: a bipartite pure state is said to be entangled if its Schmidt number is greater than one. Because particles in an entanglement do not have states, or even some properties, independently of each other this means when two-particles are so entangled, neither particle separately has a state. One must thus consider a two-particle system as a simple entity.

According to the original idea of EPR that the relative coordinate operator of the two particles commutes with their total momentum operator, $[Q_1 - Q_2, P_1 + P_2] = 0$, we can set up the entangled state representation of a continuous variable in two-mode Fock space [37], which is the common eigenvector of $Q_1 - Q_2$ and $P_1 + P_2$, i.e.

$$|\eta\rangle = \exp[-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger]|00\rangle,$$
$$\eta = \eta_1 + \mathrm{i}\eta_2, \quad (137)$$

where the subscript $i = 1, 2$ denotes the a_1-mode (a_2-mode) Fock space. Using $[a_i, a_j^\dagger] = \delta_{ij}$, we show

$$(a_1 - a_2^\dagger)|\eta\rangle = \eta|\eta\rangle, \qquad (a_2 - a_1^\dagger)|\eta\rangle = -\eta^*|\eta\rangle. \quad (138)$$

η is a complex number whose real and imaginary parts are indeed the eigenvalues of $Q_1 - Q_2$ and $P_1 + P_2$ respectively, i.e.

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \qquad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle. \quad (139)$$

By using $|00\rangle\langle 00| = :\mathrm{e}^{-a_1^\dagger a_1 - a_2^\dagger a_2}:$, and the IWOP technique we can prove the complete relation of $|\eta\rangle$,

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = \int \frac{\mathrm{d}^2\eta}{\pi} \times :\mathrm{e}^{-|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger + \eta^* a_1 - \eta a_2 + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2}: = 1, \quad (140)$$

The orthonormal property of $|\eta\rangle$ is

$$\langle\eta|\eta'\rangle = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*). \quad (141)$$

The Schmidt decomposition of $|\eta\rangle$ is

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_2\eta_1}\int_{-\infty}^{\infty}\mathrm{d}x\, |q\rangle_1 \otimes |q - \sqrt{2}\eta_1\rangle_2\, \mathrm{e}^{\mathrm{i}\sqrt{2}\eta_2 x}. \quad (142)$$

The $|\eta\rangle$ state can also be Schmidt decomposed in momentum eigenvector space as

$$|\eta\rangle = \mathrm{e}^{\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}p\, |p\rangle_a \otimes |\sqrt{2}\eta_2 - p\rangle_b \mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p}. \quad (143)$$

We call $|\eta\rangle$ the entangled state representation with a continuous variable, because the two-mode squeezing operator has its natural representation in the $\langle\eta|$ basis [38]

$$\int \frac{\mathrm{d}^2\eta}{\pi\mu}|\eta/\mu\rangle\langle\eta| = \mathrm{e}^{a_1^\dagger a_2^\dagger \tanh\lambda}\mathrm{e}^{(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)\ln \operatorname{sech}\lambda} \times \mathrm{e}^{-a_1 a_2 \tanh\lambda}, \qquad \mu = \mathrm{e}^\lambda. \quad (144)$$

The proof of (144) is found by virtue of the IWOP technique

$$\begin{aligned}\int \frac{\mathrm{d}^2\eta}{\pi\mu}|\eta/\mu\rangle\langle\eta| &= \int \frac{\mathrm{d}^2\eta}{\pi\mu}:\exp\left\{-\frac{|\eta|^2}{2}\left(1 + \frac{1}{\mu^2}\right) + \eta\left(\frac{a_1^\dagger}{\mu} - a_2\right) + \eta^*\left(a_1 - \frac{a_2^\dagger}{\mu}\right) + a_1^\dagger a_2^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right\}: \\ &= \frac{2\mu}{1+\mu^2}:\exp\left\{\frac{\mu^2}{1+\mu^2}\left(\frac{a_1^\dagger}{\mu} - a_2\right)\left(a_1 - \frac{a_2^\dagger}{\mu}\right) - (a_1 - a_2^\dagger)(a_1^\dagger - a_2)\right\}: \\ &= \mathrm{e}^{a_1^\dagger a_2^\dagger \tanh\lambda}\mathrm{e}^{(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)\ln \operatorname{sech}\lambda}\mathrm{e}^{-a_1 a_2 \tanh\lambda} \equiv S_2,\end{aligned} \quad (145)$$

and the two-mode squeezed state itself is an entangled state which entangles the idle mode and signal mode as an outcome of a parametric down-conversion process [39]. Comparing (145) with the single-mode squeezing operator $\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\sqrt{\mu}}|\frac{q}{\mu}\rangle\langle q|$ in (29) we again see the beauty and succinctness of Dirac's symbolic method. Further we see that $|\eta\rangle$ can be expressed as

$$|\eta\rangle = D(\eta)\,\mathrm{e}^{a_1^\dagger a_2^\dagger}|00\rangle, \qquad D(\eta) = \exp(\eta a^\dagger - \eta^* a). \quad (146)$$

From [40] and [41] we know that when the symmetric 50:50 beam splitter (without loss and phase shift) operating on a pair of incoming modes—one is the zero-momentum eigenstate $|p = 0\rangle_1$ (maximum squeezing in the p-direction) and the other is the zero-position eigenstate $|x = 0\rangle_2$ (maximum squeezing in x-direction)—the outgoing state is a bipartite entangled state, i.e.

$$\exp[-\pi(a_1^\dagger a_2 - a_2^\dagger a_1)/4]|p = 0\rangle_1 \otimes |x = 0\rangle_2 = \exp[a_1^\dagger a_2^\dagger]|00\rangle. \quad (147)$$

Then making a local oscillator displacement $D(\eta) = \exp[\eta a_1^\dagger - \eta^* a_1]$ for $\exp[a_1^\dagger a_2^\dagger]|00\rangle$, the state $|\eta\rangle$ is obtained. In [40] Kim *et al* also considered the operation of a beam splitter on some other incoming states, the operation is an $SU(2)$ rotation. Here we consider a more complicated problem—two-mode squeezing on a two-mode number state—and show how the entangled state representation of S_2 can solve this problem directly and exhibit the result more physically.

By using the generating function formula of two-variable Hermitian polynomials [42]

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\eta, \eta^*) = \exp(-tt' + t\eta + t'\eta^*), \quad (148)$$

where $H_{m,n}(\eta, \eta^*)$ is two-variable Hermitian polynomial

$$H_{m,n}(\eta, \eta^*) = \sum_{l=0}^{\min(m,n)} \frac{m!n!}{l!(m-l)!(n-l)!}(-)^l \eta^{m-l}\eta^{*n-l} = \frac{\partial^{n+m}}{\partial t^m\,\partial t'^n} \exp(-tt' + t\eta + t'\eta^*)\bigg|_{t,t'=0}. \quad (149)$$

we see that $|\eta\rangle$ can be represented in the two-mode number state as

$$|\eta\rangle = \sum_{m,n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|\eta|^2} H_{m,n}(\eta, \eta^*) \frac{(-)^n}{\sqrt{m!n!}}|mn\rangle. \quad (150)$$

Now we calculate the two-mode squeezed number state by using (145) and (150) and $\mu = \mathrm{e}^\lambda$,

$$\begin{aligned}
S_2|mn\rangle &= \int \frac{\mathrm{d}^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|m, n\rangle \\
&= \int \frac{\mathrm{d}^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle \mathrm{e}^{-\frac{1}{2}|\eta|^2}\frac{(-)^n}{\sqrt{n!m!}} H_{n,m}(\eta, \eta^*) \\
&= \frac{\partial^{m+n}}{\partial t^n\,\partial t'^m}\int \frac{\mathrm{d}^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle \mathrm{e}^{-\frac{1}{2}|\eta|^2}\frac{(-)^n}{\sqrt{m!n!}} \\
&\quad\times \exp(-tt' + t\eta + t'\eta^*)|_{t,t'=0} \\
&= \frac{(-)^n}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^n\,\partial t'^m}\int \frac{\mathrm{d}^2\eta}{\mu\pi}\exp\left[-\frac{1}{2}|\eta|^2\left(\frac{1}{\mu^2} + 1\right)\right. \\
&\quad\left. + \eta\left(t + \frac{a_1^\dagger}{\mu}\right) + \eta^*\left(t' - \frac{a_2^\dagger}{\mu}\right) - tt' + a_1^\dagger a_2^\dagger\right]|00\rangle\bigg|_{t,t'=0} \\
&= \frac{2\mu(-)^n}{(\mu^2+1)\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^n\,\partial t'^m}\exp\left[\frac{2\mu^2}{\mu^2+1}\left(t + \frac{a_1^\dagger}{\mu}\right)\right. \\
&\quad\left.\times\left(t' - \frac{a_2^\dagger}{\mu}\right) - tt' + a_1^\dagger a_2^\dagger\right]|00\rangle\bigg|_{t,t'=0} \\
&= \frac{2\mu(-)^n}{(\mu^2+1)\sqrt{m!n!}}\frac{\partial^n}{\partial t^n}\left[\frac{2\mu}{\mu^2+1}a_1^\dagger + \frac{\mu^2-1}{\mu^2+1}t\right]^m \\
&\quad\times\exp\left[-\frac{2\mu}{(\mu^2+1)}ta_2^\dagger + \frac{\mu^2-1}{\mu^2+1}a_2^\dagger a_1^\dagger\right]|00\rangle\bigg|_{t=0} \\
&= \frac{2\mu\sqrt{m!n!}}{(\mu^2+1)}\sum_{l=0}^{\min(m,n)}\frac{1}{l!(n-l)!(m-l)!}\left(\frac{2\mu}{\mu^2+1}a_2^\dagger\right)^{n-l} \\
&\quad\times\left(\frac{2\mu}{\mu^2+1}a_1^\dagger\right)^{m-l}\left(\frac{1-\mu^2}{\mu^2+1}\right)^l\exp\left(\frac{\mu^2-1}{\mu^2+1}a_2^\dagger a_1^\dagger\right)|00\rangle \\
&= \frac{\sqrt{m!n!}}{\cosh\lambda}\sum_{j=\max(0,n-m)}^{n}\frac{1}{j!(n-j)!(m-n+j)!} \\
&\quad\times\left(\frac{1}{\cosh\lambda}a_2^\dagger\right)^j\left(\frac{1}{\cosh\lambda}a_1^\dagger\right)^{m-n+j}(-\tanh\lambda)^{n-j} \\
&\quad\times\exp(\tanh\lambda a_2^\dagger a_1^\dagger)|00\rangle, \qquad (151)
\end{aligned}$$

which turns out to be two-mode photon-number excitation on a two-mode squeezed vacuum state $\frac{1}{\cosh\lambda}\exp(\tanh\lambda a_2^\dagger a_1^\dagger)|00\rangle$. In addition, we think that this result is relatively more compact because the exponential factor $\exp(\tanh\lambda a_2^\dagger a_1^\dagger)$ is not involved in the summation. Besides, it is encouraging and instructive that $|\eta\rangle$ also provides the Noh–Fougères–Mandel (NFM) [43, 44] operational phase operator

$$\sqrt{\frac{a_1 - a_2^\dagger}{a_1^\dagger - a_2}} \equiv \mathrm{e}^{\mathrm{i}\Phi} \quad (152)$$

with a diagonalized basis, i.e. from (138) and (140) we see that in the $\langle\eta|$ representation $\mathrm{e}^{\mathrm{i}\Phi}$ behaves as

$$\mathrm{e}^{\mathrm{i}\Phi} = \int \frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{\mathrm{i}\varphi}|\eta\rangle\langle\eta|, \qquad \mathrm{e}^{\mathrm{i}\varphi} = \left(\frac{\eta}{\eta^*}\right)^{\frac{1}{2}}, \quad (153)$$

which manifestly exhibits its phase behaviour. Note that $[a_1 - a_2^\dagger, a_1^\dagger - a_2] = 0$, so they can reside into the same square root. $\mathrm{e}^{\mathrm{i}\Phi}$ is a unitary operator. Recall that the annihilation operator a_1 can be made subject to polar decomposition as $a_1 = \sqrt{a_1^\dagger a_1 + 1}\hat{\mathrm{e}}^{\mathrm{i}\theta}$, where is the Susskind–Glogower [45] Q.7 phase operator, we may make the following decomposition

$$a_1 - a_2^\dagger = \sqrt{A^\dagger A}\mathrm{e}^{\mathrm{i}\Phi}, \qquad a_1^\dagger - a_2 = \mathrm{e}^{-\mathrm{i}\Phi}\sqrt{A^\dagger A}. \quad (154)$$

We name the correlative amplitude operator

$$A^\dagger A = (a_1^\dagger - a_2)(a_1 - a_2^\dagger). \quad (155)$$

From (138) we see

$$A^\dagger A|\eta\rangle = |\eta|^2|\eta\rangle, \quad (156)$$

so $|\eta\rangle$ is also the common eigenvector of $A^\dagger A$ and, in this sense, we say that it also exhibits the entanglement in respect of the correlative amplitude and the operational phase $\mathrm{e}^{\mathrm{i}\Phi}$ [46].

Remarkably, the simultaneous eigenstate $|\zeta\rangle$ of two commutative operators $(Q_1 + Q_2, P_1 - P_2)$ in two-mode Fock space can also be explicitly constructed [47]:

$$|\zeta\rangle = \exp[-\tfrac{1}{2}|\zeta|^2 + \zeta a_1^\dagger + \zeta^* a_2^\dagger - a_2^\dagger a_1^\dagger]|00\rangle, \qquad \zeta = \zeta_1 + \mathrm{i}\zeta_2. \quad (157)$$

The overlap between $\langle\eta|$ and $|\zeta\rangle$ is

$$\langle\eta|\zeta\rangle = \tfrac{1}{2}\exp[(\zeta\eta^* - \eta\zeta^*)/2]. \tag{158}$$

In the $\langle\eta|$ representation the two-mode Wigner operator has a compact form

$$\Delta_{12}(\rho,\gamma) \equiv \int \frac{\mathrm{d}^2\eta}{\pi^3}\,|\rho-\eta\rangle\langle\rho+\eta|\exp(\eta\gamma^* - \eta^*\gamma), \tag{159}$$

where

$$\rho = \alpha - \beta^*, \qquad \gamma = \alpha + \beta^*. \tag{160}$$

In fact, using (137), (45) and the IWOP technique we can prove

$$\begin{aligned}\Delta_{12}(\rho,\gamma) &\equiv \int \frac{\mathrm{d}^2\eta}{\pi^3}{:}\exp\{-|\eta|^2 - |\rho|^2 + (\rho-\eta)a_1^\dagger \\ &\quad - (\rho^*-\eta^*)a_2^\dagger + (\rho^*+\eta^*)a_1 + \eta\gamma^* - \eta^*\gamma \\ &\quad - (\rho+\eta)a_2 + a_2^\dagger a_1^\dagger + a_1a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\}{:} \\ &= \pi^2{:}\exp\{-|\gamma|^2 - |\rho|^2 + \gamma(a_1^\dagger + a_2) + \gamma^*(a_2^\dagger + a_1) \\ &\quad + \rho(a_1^\dagger - a_2) + \rho^*(a_1 - a_2^\dagger) - 2a_1^\dagger a_1 - 2a_2^\dagger a_2\}{:} \\ &= \pi^{-2}{:}\exp[-2(\alpha^* - a_1^\dagger)(\alpha - a_1) \\ &\quad - 2(\beta^* - a_2^\dagger)(\beta - a_2)]{:} = \Delta_1(\alpha,\alpha^*)\Delta_2(\beta,\beta^*).\end{aligned} \tag{161}$$

Using (159) and (141) we can immediately obtain the Wigner function of $|\eta\rangle$

$$\begin{aligned}&\langle\eta|\,\Delta_{12}(\rho,\gamma)|\eta\rangle \\ &= \int \frac{\mathrm{d}^2\eta'}{\pi^3}\,\langle\eta|\rho-\eta'\rangle\langle\rho+\eta'|\eta\rangle\exp(\eta'\gamma^* - \eta'^*\gamma) \\ &= \int \frac{\mathrm{d}^2\eta'}{\pi^3}\,\delta^{(2)}(\eta-\rho+\eta')\delta^{(2)}(\rho+\eta'-\eta) \\ &\quad \times \exp(\eta'\gamma^* - \eta'^*\gamma) \\ &= (4\pi)^{-1}\delta[\eta_1 - \mathrm{Re}(\alpha-\beta^*)]\delta[\eta_2 - \mathrm{Im}(\alpha-\beta^*)].\end{aligned} \tag{162}$$

Let $\alpha = \frac{1}{\sqrt{2}}(q_1 + \mathrm{i}p_1)$, $\beta = \frac{1}{\sqrt{2}}(q_2 + \mathrm{i}p_2)$, then (162) becomes

$$\begin{aligned}\langle\eta|\,\Delta_{12}(\rho,\gamma)|\eta\rangle &= (2\pi)^{-1}\,\delta[\sqrt{2}\eta_1 - (q_1 - q_2)] \\ &\quad\times\delta[\sqrt{2}\eta_2 - (p_1 + p_2)],\end{aligned} \tag{163}$$

as expected. Using (159), (114) and (145) we can directly obtain the Wigner function of the two-mode squeezed state,

$$\begin{aligned}\langle 00|S_2^{-1}\,\Delta_{12}(\rho,\gamma)S_2|00\rangle &= \int \frac{\mathrm{d}^2\eta''}{\pi\mu}\,\langle\eta''/\mu| \\ &\quad\times\int\frac{\mathrm{d}^2\eta}{\pi^3}\,|\rho-\eta\rangle\langle\rho+\eta|\int\frac{\mathrm{d}^2\eta'}{\pi\mu}\,|\eta'/\mu\rangle \\ &\quad\times\exp\left(-\frac{|\eta'|^2 + |\eta''|^2}{2} + \eta\gamma^* - \eta^*\gamma\right) \\ &= \pi^{-2}\exp(-\mu^2|\rho|^2 - |\gamma|^2/\mu^2).\end{aligned} \tag{164}$$

14. The IWOP technique for nonlinear Bose operators

In recent years nonlinear coherent states (NCS) have been given much attention because they exhibit nonclassical features, and many quantum optical states [48–50], such as the squeezed state, phase states and the negative binomial state, can be viewed as types of nonlinear coherent state [51–53]. A class of NCS can be realized physically as the stationary states of centre-of-mass motion of a trapped ion [54]. The nonlinear coherent state $|z\rangle_f$ is defined as eigenstate of $f(N)a$,

$$f(N)a|z\rangle_f = z|z\rangle_f, \tag{165}$$

where $f(N)$ is an operator-valued function of the number operator $N = a^\dagger a$, satisfying

$$\left[f(N)a, \frac{1}{f(N-1)}a^\dagger\right] = 1, \tag{166}$$

and

$$f(N)a = af(N-1), \qquad \frac{1}{f(N-1)}a^\dagger = a^\dagger\frac{1}{f(N)}. \tag{167}$$

It is seen that $|z\rangle_f$ has the form [48–53],

$$\begin{aligned}|z\rangle_f &= \exp\left[\frac{z}{f(N-1)}a^\dagger\right]|0\rangle \\ &= |0\rangle + \sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}\prod_{m=0}^{n-1}f(m)}|n\rangle,\end{aligned} \tag{168}$$

where $|0\rangle$ is the vacuum state in Fock space. Note that if the state $|z\rangle_f$ is normalizable, the sum

$$1 + \sum_{n=1}^{\infty}|z|^{2n}\Big/\left\{n!\left|\prod_{m=0}^{n-1}f(m)\right|^2\right\} \tag{169}$$

must be convergent.

The completeness relation for the nonlinear coherent state can also be obtained by introducing the corresponding IWOP technique. Note

$$f(N)a\frac{1}{f(N-1)}a^\dagger = aa^\dagger, \tag{170}$$

then we see that $f(N)a$ and $\frac{1}{f(N-1)}a^\dagger$ behave like a and $a^\dagger$ respectively, so we introduce the generalized normal ordering symbol $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$ for $f(N)a$ and $\frac{1}{f(N-1)}a^\dagger$. When all $\frac{1}{f(N-1)}a^\dagger$ stand on the left of $f(N)a$, we say that they are in generalized normal ordering. Let us outline the IWOP technique for nonlinear Bose operator partners $f(N)a$ and $\frac{1}{f(N-1)}a^\dagger$ [55]:

(1) The order of nonlinear Bose operators $f(N)a$ and $\frac{1}{f(N-1)}a^\dagger$ within a generalized normally ordered product $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$ can be permuted. That is to say, even though $[f(N)a, \frac{1}{f(N-1)}a^\dagger] = 1$, we can have

$$\begin{aligned}\genfrac{}{}{0pt}{}{\circ}{\circ}f(N)a\frac{1}{f(N-1)}a^\dagger\genfrac{}{}{0pt}{}{\circ}{\circ} &= \genfrac{}{}{0pt}{}{\circ}{\circ}\frac{1}{f(N-1)}a^\dagger f(N)a\genfrac{}{}{0pt}{}{\circ}{\circ} \\ &= \frac{1}{f(N-1)}a^\dagger f(N)a.\end{aligned} \tag{171}$$

(2) The symbol $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$ which is within another symbol $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$ can be deleted.

(3) A generalized normally ordered product can be integrated or differentiated with respect to a c-number provided the integration is convergent.

(4) The vacuum state projection operator in the generalized normal ordering form is

$$|0\rangle\langle 0| = \genfrac{}{}{0pt}{}{\circ}{\circ}\exp\left(-\frac{1}{f(N-1)}a^\dagger f(N)a\right)\genfrac{}{}{0pt}{}{\circ}{\circ}. \tag{172}$$

To prove this we turn the usual normal ordering of $|0\rangle\langle 0|$ to generalized normal ordering, using

$$:(a^\dagger a)^n: = a^{\dagger n}a^n = \left[\frac{1}{f(N-1)}a^\dagger\right]^n [f(N)a]^n$$

$$= {}^{\circ}_{\circ}\left[\frac{1}{f(N-1)}a^\dagger f(N)a\right]^n {}^{\circ}_{\circ}, \tag{173}$$

and we have

$$|0\rangle\langle 0| = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}{}^{\circ}_{\circ}\left[\frac{1}{f(N-1)}a^\dagger f(N)a\right]^n {}^{\circ}_{\circ}$$

$$= {}^{\circ}_{\circ}\exp\left(-\frac{1}{f(N-1)}a^\dagger f(N)a\right){}^{\circ}_{\circ}. \tag{174}$$

To make up a completeness relation of nonlinear coherent states we introduce another state

$$|z\rangle\rangle_f = \exp[za^\dagger f(N)]|0\rangle. \tag{175}$$

In similar to equation (168), $|z\rangle\rangle_f$ has physical meaning unless the sum

$$1 + \sum_{n=1}^{\infty}|z|^{2n}\left|\prod_{m=0}^{n-1} f(m)\right|^2 \bigg/ n! \tag{176}$$

is convergent. We can easily verify that $|z\rangle\rangle_f$ satisfies the equation

$$\frac{1}{f(N)}a|z\rangle\rangle_f = z|z\rangle\rangle_f. \tag{177}$$

The overlap between $(|z'\rangle\rangle_f)^\dagger = \langle 0|\exp[z'^* f(N)a]$ and $|z\rangle_f$ is

$${}_f\langle\langle z'|z\rangle_f = \langle 0|\exp[z'^* f(N)a]\exp\left[\frac{z}{f(N-1)}a^\dagger\right]|0\rangle$$

$$= \exp(z'^* z), \tag{178}$$

which indicates that NCS is not an orthonormal state. The completeness relation of NCS then follows:

$$\int\frac{d^2z}{\pi}e^{-|z|^2}|z\rangle_f\,{}_f\langle\langle z| = \int\frac{d^2z}{\pi}\exp\left[-|z|^2 + \frac{z}{f(N-1)}a^\dagger\right]$$

$$\times {}^{\circ}_{\circ}\exp\left[-\frac{1}{f(N-1)}a^\dagger f(N)a\right]{}^{\circ}_{\circ}\exp[z^* f(N)a], \tag{179}$$

where $e^{-|z|^2}$ is a integration measure for convergence. Looking at (179), we see that $\exp[\frac{z}{f(N-1)}a^\dagger]$ is on the left of ${}^{\circ}_{\circ}\exp[-\frac{1}{f(N-1)}a^\dagger f(N)a]{}^{\circ}_{\circ}$, while $\exp[z^* f(N)a]$ is on its right-hand side, thus the three exponentials are automatically in generalized normal ordering and can be combined into a unique exponential within ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$, i.e.

$$\int\frac{d^2z}{\pi}e^{-|z|^2}|z\rangle_f\,{}_f\langle\langle z| = \int\frac{d^2z}{\pi}{}^{\circ}_{\circ}\exp[-|z|^2 + \frac{z}{f(N-1)}a^\dagger$$

$$- \frac{1}{f(N-1)}a^\dagger f(N)a + z^* f(N)a]{}^{\circ}_{\circ} = 1, \tag{180}$$

which is valid for all well-defined operator functions $f(N)$. Equation (180) is the over-completeness relation for nonlinear coherent states, its conjugate is

$$\int\frac{d^2z}{\pi}e^{-|z|^2}|z\rangle\rangle_f\,{}_f\langle z| = 1. \tag{181}$$

It is remarkable that in (181) the ket and bra are not mutually Hermitian conjugate. It must also be emphasized that within the symbol ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ operator $\frac{1}{f(N-1)}a^\dagger$ commutes with $f(N)a$, which means that within ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ they can both be considered as c-number integral parameters while the integration over d^2z is being performed.

However, we must notice that although the integration value of (180) (and (181)) gives unity, the resolution of ${}^{\circ}_{\circ}\exp[-|z|^2 + \frac{z}{f(N-1)}a^\dagger - \frac{1}{f(N-1)}a^\dagger f(N)a + z^* f(N)a]{}^{\circ}_{\circ}$ into physically meaningful ket–bra states requires caution. This is because not every NCS, as defined by (168) and (175), is normalizable. Some NCS are physically meaningful only for some restricted value of z (see the convergent conditions (169) and (176).

Using the IWOP and (181) we can derive some operator formulae about nonlinear Bose operators, for example

$$e^{\lambda[f(N)a]^2}e^{\sigma[\frac{1}{f(N-1)}a^\dagger]^2} = \int\frac{d^2z}{\pi}e^{\lambda z^2}|z\rangle_f\,{}_f\langle\langle z|e^{\sigma z^{*2}}$$

$$= \int\frac{d^2z}{\pi}{}^{\circ}_{\circ}\exp[-|z|^2 + \lambda z^2 + \sigma z^{*2}$$

$$+ \frac{z}{f(N-1)}a^\dagger - \frac{1}{f(N-1)}a^\dagger f(N)a + z^* f(N)a]{}^{\circ}_{\circ}$$

$$= \frac{1}{\sqrt{1-4\lambda\sigma}}\exp\left\{\sigma\left[\frac{1}{f(N-1)}a^\dagger\right]^2 \bigg/ (1-4\lambda\sigma)\right\}$$

$$\times \exp[-a^\dagger a\ln(1-4\lambda\sigma)]\exp\{\lambda[f(N)a]^2/(1-4\lambda\sigma)\}. \tag{182}$$

When $f(N) = 1$, equation (182) reduces to (58).

Now we construct the following ket–bra projection operator in integral form and using the generalized IWOP technique to carry out the integral,

$$U = s^{*-1/2}|s|\int_{-\infty}^{\infty}\frac{d^2z}{\pi}|sz - rz^*\rangle_f\,{}_f\langle\langle z|$$

$$= s^{*-1/2}|s|\int_{-\infty}^{\infty}\frac{d^2z}{\pi}{}^{\circ}_{\circ}\exp\left\{-|s|^2|z|^2\right.$$

$$+ sz\frac{1}{f(N-1)}a^\dagger + z^*\left[f(N)a - r\frac{1}{f(N-1)}a^\dagger\right]$$

$$\left.+ \frac{1}{2}r^* sz^2 + \frac{1}{2}rs^* z^{*2} - \frac{1}{f(N-1)}a^\dagger f(N)a\right\}{}^{\circ}_{\circ}$$

$$= \exp\left\{-\frac{r}{2s^*}\left[\frac{1}{f(N-1)}a^\dagger\right]^2\right\}\exp[-(a^\dagger a + 1/2)\ln s^*]$$

$$\times \exp\left\{\frac{r^*}{2s^*}[f(N)a]^2\right\}, \tag{183}$$

where s and r satisfy relation $|s|^2 - |r|^2 = 1$ and we have used the operator identity

$$e^{ka^\dagger a} = :\exp\{(e^k - 1)a^\dagger a\}:$$

$$= {}^{\circ}_{\circ}\exp\left\{(e^k - 1)\frac{1}{f(N-1)}a^\dagger f(N)a\right\}{}^{\circ}_{\circ}. \tag{184}$$

U is a similar transformation operator, only when $f = 1$, U is unitary (due to the existence of $s^{*-1/2}|s|$ in equation (183)). U engenders a similar transformation for the nonlinear Bose operator

$$U^{-1}f(N)aU = sf(N)a - r\frac{1}{f(N-1)}a^\dagger,$$

$$U^{-1}\frac{1}{f(N-1)}a^\dagger U = s^*\frac{1}{f(N-1)}a^\dagger - r^* f(N)a. \tag{185}$$

We call U the generalized single-mode squeezing operator. It is interesting to observe that the operators appearing on the right-hand side of (183) satisfy

$$\left[(f(N)a)^2, \left(\frac{1}{f(N-1)}a^\dagger\right)^2\right] = 4(a^\dagger a + 1/2),$$
$$[(f(N)a)^2, a^\dagger a] = 2(f(N)a)^2, \tag{186}$$
$$\left[\left(\frac{1}{f(N-1)}a^\dagger\right)^2, a^\dagger a\right] = -2\left(\frac{1}{f(N-1)}a^\dagger\right)^2$$

which means they also make up an $SU(1,1)$ Lie algebra.

15. Concluding remarks

In summary, we have followed the guidance of Dirac to further understand his symbolic method. We find that the introduction of IWOP can well deal with a lot of integral-form nonsymmetric ket–bra projection operators, which manifestly show the correspondence between unitary operators and c-number classical transforms. The IWOP technique renders Dirac's representation theory more powerful and applicable, and operator ordering (normally ordered, antinormally ordered and Weyl ordered) problems can be tackled in a unified and efficient way. In particular, the normally ordered generalized squeezing operators can be derived without appealing to the usual Lie algebraic method. The antinormally ordered and Weyl ordered expansions of density matrices are obtained. Many new quantum mechanical representations which possess completeness relations can be derived by virtue of the IWOP technique. The completeness relation of the entangled state of continuum variables is easily proved. The IWOP technique can be generalized to the nonlinear coherent state as well. Therefore, the IWOP technique accompanies Dirac's symbolic method naturally and is a profound mathematical basis for quantum optics theory. Many useful operator identities are deduced by the IWOP technique. For readers' convenience, here we list some useful operator identities: equations (31)–(33), (38), (40), (44), (45), (54), (59), (62)–(64), (79), (81), (84), (97), (101), (103), (110), (112), (117), (118), (123), (132), (140), (145), (150), (151), (153), (159), (180), (183).

Besides, the IWOP can also be extended to the fermionic case and the spherical coordinate system, and be applied to the study of group representation theory, generalized coherent state theory and solving some dynamic problems.

Let us quote Einstein: 'In the science of physics, the way to the more deeply elementary knowledge is connected with the most precise mathematical methods'. Q.8

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