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A study on one-dimensional nonlinear consolidation of double-layered soil

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Abstract

In this paper, an analytical solution is derived for one-dimensional nonlinear consolidation of double-layered soil, based on the assumptions proposed by Davis and Raymond (Davis EH, Raymond GP. A non-linear theory of consolidation. Geotechnique 1965;15(2):161–73) that the decrease in permeability is proportional to the decrease in compressibility during the consolidation of the soil and the distribution of initial effective pressures does not vary with depth. The analytical solutions so far available for nonlinear consolidation are all special cases of this solution. Using the solutions obtained, some diagrams were prepared and the relevant consolidation behavior of layered soils is discussed. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Double-layered soil; Non-linear consolidation theory; Analytical solution; Time-dependent loading

1. Introduction

Since Terzaghi published his consolidation theory and the principle of effective stress, research work on consolidation problems has greatly increased. The conventional consolidation theories often neglected the non-linearity of soil for practical purposes. Studies of nonlinear consolidation behavior of soft soil started from about 40 years ago [1–10]. Many efforts have been made to obtain analytical solutions for different kinds of one-dimensional consolidation theories. Davis and Raymond [2] derived firstly an analytical solution for the constant loading case based on the

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assumptions that the decrease in permeability is proportional to the decrease in compressibility during the consolidation of a soil and the distribution of initial effective pressures is constant with depth. With the same assumption about the compressibility and permeability of a soil, Xie et al. [9] developed an analytical solution to the one dimensional consolidation problem for time-dependent loading. The solution given by Davis and Raymond was regarded as a special case of it. Poskitt [4] studied a more general one-dimensional nonlinear consolidation problem by using a perturbation method, but no explicit solution was given. For the linear consolidation of layered soils, some analytical solutions have been reported [11–16].

However, there seems to be no analytical solution for nonlinear consolidation taking the layered characteristics of soil into consideration. This may be mainly due to mathematical difficulty. In this paper, an analytical solution is derived for one-dimensional nonlinear consolidation of double-layered soil considering time-dependent loading, based on the same assumptions proposed by Davis and Raymond [2] except for loading condition, and all the analytical solutions so far developed are summarized in the tables. The nonlinear consolidation behavior of double-layered soil is also discussed.

2. Mathematical modeling

The one-dimensional nonlinear consolidation of double-layered clayey soil is based on the schematic diagram shown in Fig. 1.

In Fig. 1, h_i = the thickness of the clay layer i ($i = 1, 2$); $H = h_1 + h_2$, the total thickness of two clay layers; $q(t)$ = the uniformly distributed load applied on the top surface of the soil, is shown in Fig. 2, in which, q_0, q_u = initial and ultimate load respectively, and t_c = the construction time.

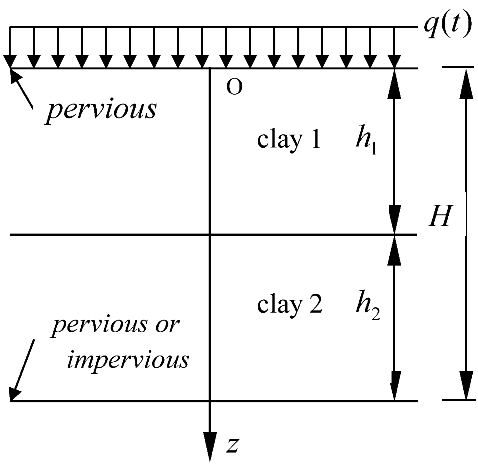

Fig. 1. Double-layered clayey soil.

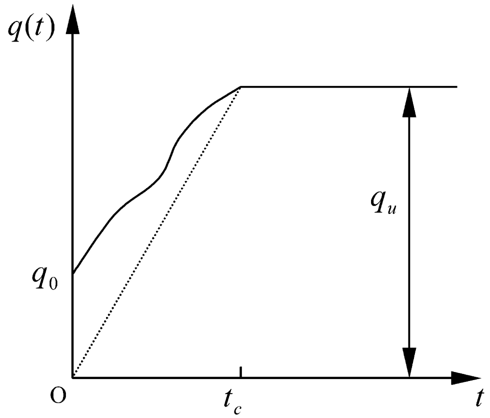

Fig. 2. Loading curve.

Adopting all the hypothesis of one-dimensional nonlinear consolidation theory by Davis and Raymond [2], except for the constant loading assumption, we can obtain the differential governing equations for the consolidation of each clay layer as follows:

$$c_{vi}\left[\frac{\partial^2 u_i}{\partial z^2}+\frac{1}{\sigma'_i}\left(\frac{\partial u_i}{\partial z}\right)^2\right]=\left(\frac{\partial u_i}{\partial t}-\frac{dq}{dt}\right)\quad (i=1,2) \tag{1}$$

where c_{vi}, u_i, σ'_i = the coefficient of consolidation, the excess pore water pressure and the effective pressure (i.e. effective vertical stress) in layer i respectively. t, z = the variables of time and space respectively.

According to the assumption that the decrease in permeability is proportional to the decrease in compressibility [2], c_{vi} is given by $c_{vi}=k_{v0i}/(m_{v0i}\gamma_w)$, in which, γ_w the unit weight of water; $m_{v0i}=\frac{0.434C_{ci}}{(1+e_{0i})\sigma'_{0i}}$, the initial coefficient of volume compressibility of the clay in layer i; k_{v0i}, C_{ci} = the initial vertical coefficient of permeability and the compression index in layer i; e_{0i} = the initial void ratio in layer i corresponding to the initial effective pressure σ'_{0i}. Since initial effective pressure is assumed to be constant [2], $\sigma'_{01}=\sigma'_{02}=\sigma'_0$.

According to Terzaghi's principle of effective stress, σ'_i can be expressed as:

$$\sigma'_i=q+\sigma'_{0i}-u_i=q+\sigma'_0-u_i \tag{1a}$$

Let

$$\omega_i=\ln\frac{\sigma'_i}{\sigma'_{0i}+q} \tag{2}$$

Eq. (1) can be simplified in terms of function ω_i as follows:

$$\frac{\partial\omega_i}{\partial t}=c_{vi}\frac{\partial^2\omega_i}{\partial z^2}+R(t)\quad (i=1,2) \tag{3}$$

in which

$$R(t) = -\frac{1}{\sigma'_0 + q}\frac{\mathrm{d}q}{\mathrm{d}t} \tag{3a}$$

The boundary conditions corresponding to Eq. (3) can be expressed by the following equations:

$$z = 0 : u_1 = 0, \text{ or } \omega_1 = 0 \tag{4}$$

$$z = H : k_{v2}\frac{\partial u_2}{\partial z} = 0, \text{ or } \frac{\partial \omega_2}{\partial z} = 0 \;\; \text{(for single-drainage situation)} \tag{5a}$$

$$u_2 = 0, \text{ or } \omega_2 = 0 \;\; \text{(for double-drainage situation)} \tag{5b}$$

$$z = h_1 : u_1 = u_2, \text{ or } \omega_1 = \omega_2 \tag{6}$$

$$k_{v1}\frac{\partial u_1}{\partial z} = k_{v2}\frac{\partial u_2}{\partial z}, \text{ or } \frac{\partial \omega_1}{\partial z} = K\frac{\partial \omega_2}{\partial z} \tag{7}$$

where $K = \frac{c_{v2}m_{v02}}{c_{v1}m_{v01}} = \frac{k_{v02}}{k_{v01}}$.

$$t = 0 : u_i = q(0) = q_0, \text{ or } \omega_i = \omega_0 \tag{8}$$

where $\omega_0 = \ln\frac{\sigma'_0}{\sigma'_0 + q_0}$.

3. Solutions

3.1. The excess pore water pressure

3.1.1. Single-drainage situation

In order to find the solution of Eq. (3) subjected to the conditions of Eqs. (4)–(8), we assume that the solution of Eq. (3) may have the form of an infinite series as follows:

$$\omega_i = \sum_{m=1}^{\infty} g_{mi}(z)\mathrm{e}^{-\beta_m t}[B_m + C_m T_m(t)] \qquad (i = 1, 2) \tag{9}$$

where

$$g_{m1}(z) = \sin(\lambda_m \frac{z}{h_1}) \tag{9a}$$

$$g_{m2}(z) = A_m \cos(\mu\lambda_m \frac{H - z}{h_1}) \tag{9b}$$

and β_m, B_m, C_m, A_m, λ_m, μ = unknown coefficients to be found; $T_m(t)$ = a function of t, resulting from the item $R(t)$ in Eq. (3).

It can be shown that Eq. (9) satisfies Eqs. (3), (4) and (5a). Using other conditions represented by Eqs. (6) to (8), we can get:

$$A_m = \begin{cases} \dfrac{\sin(\lambda_m)}{\cos(\mu c\lambda_m)} & \cos(\mu c\lambda_m) \neq 0 \\ \dfrac{1}{\mu K}\dfrac{\cos(\lambda_m)}{\sin(\mu c\lambda_m)} & \cos(\mu c\lambda_m) = 0 \end{cases} \tag{10}$$

where

$$c = h_2/h_1 \tag{10a}$$

$$\beta_m = \frac{c_{v1}\lambda_m^2}{h_1^2} \tag{10b}$$

$$T_m = \int_0^t e^{\beta_m t} R(\tau)\mathrm{d}\tau \tag{10c}$$

$$\mu = \sqrt{\frac{c_{v1}}{c_{v2}}}. \tag{10d}$$

The positive eigenvalues λ_m satisfy the following eigen-equation:

$$\mu K \sin(\mu c\lambda_m)\sin(\lambda_m) - \cos(\mu c\lambda_m)\cos(\lambda_m) = 0 \tag{11}$$

Furthermore, we can obtain the following two equations:

$$\sum_{m=1}^{\infty} C_m g_{mi}(z) = 1 \quad (i = 1, 2) \tag{12}$$

$$\sum_{m=1}^{\infty} B_m g_{mi}(z) = \omega_0 \quad (i = 1, 2) \tag{13}$$

Using the following orthogonal relation:

$$\int_0^{h_1} g_{m1}(z)g_{n1}(z)\mathrm{d}z + \mu^2 K\int_{h_1}^{H} g_{m2}(z)g_{n2}(z)\mathrm{d}z = \begin{cases} 0 & m \neq n \\ \dfrac{1}{2}h_1(1+\mu^2 KcA_m^2) & m = n \end{cases} \tag{14}$$

we can get the coefficients B_m and C_m as follows:

$$B_m = \frac{\int_0^{h_1} g_{m1}(z)\mathrm{d}z + \mu^2 K\int_{h_1}^{H} g_{m2}(z)\mathrm{d}z}{\int_0^{h_1} g_{m1}^2(z)\mathrm{d}z + \mu^2 K\int_{h_1}^{H} g_{m2}^2(z)\mathrm{d}z}\omega_0 = \frac{2\omega_0}{\lambda_m(1+\mu^2 KcA_m^2)} \tag{15}$$

$$C_m = \frac{\int_0^{h_1} g_{m1}(z)\mathrm{d}z + \mu^2 K \int_{h_1}^{H} g_{m2}(z)\mathrm{d}z}{\int_0^{h_1} g_{m1}^2(z)\mathrm{d}z + \mu^2 K \int_{h_1}^{H} g_{m2}^2(z)\mathrm{d}z} = \frac{2}{\lambda_m(1+\mu^2 KcA_m^2)} \tag{16}$$

Substituting all those coefficients into Eq. (9), we can get the final results for ω_1 and ω_2:

$$\omega_1 = \sum_{m=1}^{\infty} \sin(\lambda_m \frac{z}{h_1}) \mathrm{e}^{-\beta_m t}\left[B_m + C_m \int_0^t \mathrm{e}^{\beta_m \tau} R(\tau)\mathrm{d}\tau\right] \tag{17a}$$

$$\omega_2 = \sum_{m=1}^{\infty} A_m \cos(\mu\lambda_m \frac{H-z}{h_1}) \mathrm{e}^{-\beta_m t}\left[B_m + C_m \int_0^t \mathrm{e}^{\beta_m \tau} R(\tau)\mathrm{d}\tau\right] \tag{7b}$$

And finally, according to the substitution $\omega_i = \ln\frac{\sigma_i'}{\sigma_0'+q}$, the excess pore pressure can be expressed as:

$$u_1 = (\sigma_0' + q)(1 - \mathrm{e}^{\omega_1}) \tag{18a}$$

$$u_2 = (\sigma_0' + q)(1 - \mathrm{e}^{\omega_2}) \tag{18b}$$

3.1.2. Double-drainage situation

In a similar way to that described above, we can obtain the solution for the double-drainage situation.

Eqs. (18a) and (18b) are still valid, but ω_1 and ω_2 should be replaced by the following values:

$$\omega_1 = \sum_{m=1}^{\infty} \sin(\lambda_m \frac{z}{h_1}) \mathrm{e}^{-\beta_m t}\left[B_m + C_m \int_0^t \mathrm{e}^{\beta_m \tau} R(\tau)\mathrm{d}\tau\right] \tag{19a}$$

$$\omega_2 = \sum_{m=1}^{\infty} A_m \sin(\mu\lambda_m \frac{H-z}{h_1}) \mathrm{e}^{-\beta_m t}\left[B_m + C_m \int_0^t \mathrm{e}^{\beta_m \tau} R(\tau)\mathrm{d}\tau\right] \tag{19b}$$

where λ_m = the positive roots for the following eigen-equation:

$$\mu K \cos(\mu c\lambda_m)\sin(\lambda_m) + \sin(\mu c\lambda_m)\cos(\lambda_m) = 0 \tag{20}$$

and

$$A_m = \begin{cases} \dfrac{\sin(\lambda_m)}{\sin(\mu c\lambda_m)} & \sin(\mu c\lambda_m) \neq 0 \\[2ex] -\dfrac{1}{\mu K}\dfrac{\cos(\lambda_m)}{\cos(\mu c\lambda_m)} & \sin(\mu c\lambda_m) = 0 \end{cases} \tag{21}$$

$$B_m = \frac{2(1+\mu K A_m)\omega_0}{\lambda_m(1+\mu^2 K c A_m^2)} \tag{22}$$

$$C_m = \frac{2(1+\mu K A_m)}{\lambda_m(1+\mu^2 K c A_m^2)} \tag{23}$$

Other parameters and coefficients have the same definitions as those in the single-drainage situation.

3.2. Average degree of consolidation

Average degree of consolidation can be defined either in terms of settlement or in terms of effective stress. While the former shows the rate of settlement development, the later indicates the rate of the increase of effective pressure or the rate of the dissipation of excess pore water pressure.

3.2.1. Average degree of consolidation for each layer

The average degree of consolidation of each layer defined in terms of settlement, U_{si}, can be given by:

$$U_{si} = \frac{\int_{z_{i-1}}^{z_i} \varepsilon_i \mathrm{d}z}{\int_{z_{i-1}}^{z_i} \varepsilon_{fi} \mathrm{d}z} = \frac{\int_{z_{i-1}}^{z_i} (\mathrm{e}_{0i} - \mathrm{e}_i)\mathrm{d}z}{\int_{z_{i-1}}^{z_i} (\mathrm{e}_{0i} - \mathrm{e}_{fi})\mathrm{d}z} = \frac{\int_{z_{i-1}}^{z_i} \log(\frac{\sigma_i'}{\sigma_0'})\mathrm{d}z}{\int_{z_{i-1}}^{z_i} \log(\frac{\sigma_{fi}'}{\sigma_0'})\mathrm{d}z} \quad (i = 1, 2) \tag{24}$$

where $z_0 = 0$; $z_1 = h_1$; $z_2 = H$; $\varepsilon_i = \frac{\mathrm{e}_{0i}-\mathrm{e}_i}{1+\mathrm{e}_{0i}} = \frac{C_{ci}}{1+\mathrm{e}_{0i}}\log(\frac{\sigma_i'}{\sigma_0'})$, the vertical strain in layer i; $\varepsilon_{fi} = \frac{\mathrm{e}_{0i}-\mathrm{e}_{fi}}{1+\mathrm{e}_{0i}} = \frac{C_{ci}}{1+\mathrm{e}_{0i}}\log(\frac{\sigma_{fi}'}{\sigma_0'})$, the final vertical strain in layer i; e_{fi} =the final void ratio in layer i corresponding to the final effective pressure given by $\sigma_{fi}' = \sigma_f' = \sigma_0' + q_u$ according to Eq. (1a).

Substituting Eqs. (1a) and (18) into Eq. (24), we get

$$U_{s1} = \frac{\int_0^{h_1} \log\left(\frac{\sigma_1'}{\sigma_0'}\right)\mathrm{d}z}{\int_0^{h_1} \log\left(\frac{\sigma_f'}{\sigma_0'}\right)\mathrm{d}z} = \frac{\ln\frac{\sigma_0'+q}{\sigma_0'} + \frac{1}{h_1}\int_0^{h_1}\omega_1 \mathrm{d}z}{\ln\frac{\sigma_0'+q_u}{\sigma_0'}} \tag{24a}$$

$$U_{s2} = \frac{\int_{h_1}^{H} \log\left(\frac{\sigma_2'}{\sigma_0'}\right)\mathrm{d}z}{\int_{h_1}^{H} \log\left(\frac{\sigma_f'}{\sigma_0'}\right)\mathrm{d}z} = \frac{\ln\frac{\sigma_0'+q}{\sigma_0'} + \frac{1}{h_2}\int_{h_1}^{H}\omega_2 \mathrm{d}z}{\ln\frac{\sigma_0'+q_u}{\sigma_0'}} \tag{24b}$$

The average degree of consolidation of each layer defined in terms of effective stress, U_{pi}, can be given by

$$U_{pi}=\frac{\int_{z_{i-1}}^{z_i}(\sigma'_i-\sigma'_{0i})\mathrm{d}z}{\int_{z_{i-1}}^{z_i}(\sigma'_{\mathrm{f}i}-\sigma'_{0i})\mathrm{d}z}=\frac{\int_{z_{i-1}}^{z_i}(q-u_i)\mathrm{d}z}{\int_{z_{i-1}}^{z_i}q_\mathrm{u}\mathrm{d}z} \tag{25}$$

Thus

$$U_{\mathrm{p}1}=\frac{\int_0^{h_1}(q-u_1)\mathrm{d}z}{\int_0^{h_1}q_\mathrm{u}\mathrm{d}z}=\frac{q-\frac{1}{h_1}\int_0^{h_1}u_1\mathrm{d}z}{q_\mathrm{u}} \tag{25a}$$

$$U_{\mathrm{p}2}=\frac{\int_{h_1}^{H}(q-u_2)\mathrm{d}z}{\int_{h_1}^{H}q_\mathrm{u}\mathrm{d}z}=\frac{q-\frac{1}{h_2}\int_{h_1}^{H}u_2\mathrm{d}z}{q_\mathrm{u}} \tag{25b}$$

3.2.2. Average degree of consolidation for the whole two-layered clay stratum

The total average degree of consolidation of the whole two-layered clay stratum defined in terms of settlement, U_s, can be given by:

$$U_\mathrm{s}=\frac{\sum_{i=1}^{2}\int_{z_{i-1}}^{z_i}\varepsilon_i\mathrm{d}z}{\sum_{i=1}^{2}\int_{z_{i-1}}^{z_i}\varepsilon_{\mathrm{f}i}\mathrm{d}z}=\frac{\frac{C_{\mathrm{c}1}}{1+\mathrm{e}_{01}}\int_0^{h_1}\log(\frac{\sigma'_1}{\sigma'_0})\mathrm{d}z+\frac{C_{\mathrm{c}2}}{1+\mathrm{e}_{02}}\int_{h_1}^{H}\log(\frac{\sigma'_2}{\sigma'_0})\mathrm{d}z}{\frac{C_{\mathrm{c}1}}{1+\mathrm{e}_{01}}\int_0^{h_1}\log(\frac{\sigma'_f}{\sigma'_0})\mathrm{d}z+\frac{C_{\mathrm{c}2}}{1+\mathrm{e}_{02}}\int_{h_1}^{H}\log(\frac{\sigma'_f}{\sigma'_0})\mathrm{d}z}=\frac{U_{\mathrm{s}1}+\mu^2KcU_{\mathrm{s}2}}{1+\mu^2Kc} \tag{26}$$

Meanwhile, the total average degree of consolidation defined in terms of effective stress, U_p, can be derived as follows:

$$U_\mathrm{p}=\frac{\sum_{i=1}^{2}\int_{z_{i-1}}^{z_i}(q-u_i)\mathrm{d}z}{\sum_{i=1}^{2}\int_{z_{i-1}}^{z_i}q_\mathrm{u}\mathrm{d}z}=\frac{U_{\mathrm{p}1}+cU_{\mathrm{p}2}}{1+c} \tag{27}$$

It has been well reported [13–16] that in a linear consolidation of layered soils, there is no difference between $U_{\mathrm{s}i}$ and $U_{\mathrm{p}i}$ although U_s is deferent from U_p. However, as can be seen from Eqs. (24)–(27), in a nonlinear consolidation, not only is U_s deferent from U_p, but also $U_{\mathrm{s}i}$ is deferent from $U_{\mathrm{p}i}$.

4. Special cases

4.1. Solutions for the cases of constant loading and linear loading

Based on the above solutions, results for two kinds of loading conditions: (1) constant loading, (2) linear loading are given in explicit forms and listed in Table 1 and Table 2 respectively.

For the case of constant loading: $q(t) = q_u$, $R(t) = 0$.

And for the case of linear loading (dash line in Fig. 2) that increases linearly with time and becomes constant at time t_c, $q(t)$ can be expressed as:

$$q(t) = \begin{cases} \dfrac{t}{t_c} q_u & (t_c \geqslant t \geqslant 0) \\ q_u & (t \geqslant t_c) \end{cases} \tag{28}$$

Therefore,

$$R(t) = \begin{cases} \dfrac{-1}{t + t_c \sigma_0'/q_u} & (t_c^- \geqslant t \geqslant 0) \\ 0 & (t \geqslant t_c^+) \end{cases} \tag{28a}$$

Some parameters in Tables 1 and 2 are given as follows:

$$T_{v1} = \frac{c_{v1} t}{h_1^2} \tag{29}$$

$$T_{vc1} = \frac{c_{v1} t_c}{h_1^2} \tag{30}$$

$$N_\sigma = \frac{\sigma_f'}{\sigma_0'} = \frac{\sigma_0' + q_u}{\sigma_0'} \tag{31}$$

$$T = \frac{T_{vc1} + (N_\sigma - 1) T_{v1}}{T_{vc1}} \tag{32}$$

4.2. Solution for the case of one-layer soil

Let $\mu = K = 1$, the solution listed in Table 1 reduces to be the one given by Davis and Raymond [2], and that in Table 2 turns out to be the one given by Xie et al. [9]. These are summarized in Table 3.

It can be seen from Table 3 that the average degree of consolidation defined in terms of settlement (i.e. U_s) in the solution given by Davis and Raymond is the same as that given by Terzaghi for linear consolidation, indicating that the rate of settlement in a nonlinear consolidation subjecting to the assumptions made by Davis and Raymond is no different from that in a linear consolidation. But the excess pore water pressure u and its rate of dissipation (indicated by U_p) in such nonlinear consolidation are different from those in a linear consolidation. However, the following deduction will show that the difference will also disappear as the value of N_σ approaches to 1.

Mathematically, we can obtain

$$\lim_{N_\sigma \to 1} \frac{N_\sigma}{N_\sigma - 1}\left(1 - N_\sigma^{-B}\right) = B \tag{33}$$

Table 1
Solutions for the case of constant loading

Excess pore water pressure and average degree of consolidation

$$u_1 = \frac{q_{\mathrm{u}} N_\sigma}{\mathrm{N}_\sigma - 1}(1 - \mathrm{e}^{\omega_1}) \quad u_2 = \frac{q_{\mathrm{u}} N_\sigma}{\mathrm{N}_\sigma - 1}(1 - \mathrm{e}^{\omega_2})$$

$$U_{\mathrm{s}_1} = 1 - \sum_{m=1}^{\infty} \frac{2(1 - \cos\lambda_m)}{\lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}}$$

$$U_{\mathrm{s}2} = 1 - \sum_{m=1}^{\infty} \frac{2A_m \sin(\mu c \lambda_m)}{\mu c \lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}}$$

$$U_{\mathrm{s}} = 1 - \frac{1}{1 + \mu^2 K_c} \sum_{m=1}^{\infty} \frac{2}{\lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \quad \text{(single-drainage)}$$

$$U_{\mathrm{s}1} = 1 - \sum_{m=1}^{\infty} \frac{2(1 + \mu K A_m)(1 - \cos\lambda_m)}{\lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{\lambda_m^2 T_{\mathrm{v}1}}$$

$$U_{\mathrm{s}2} = 1 - \sum_{m=1}^{\infty} \frac{2A_m (1 + \mu K A_m)[1 - \cos(\mu c \lambda_m)]}{\mu c \lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}}$$

$$U_{\mathrm{s}} = 1 - \frac{1}{1 + \mu^2 K c} \sum_{m=1}^{\infty} \frac{2(1 + \mu K A_m)^2}{\lambda_m^2 (1 + \mu^2 K c A_m^2)} \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \quad \text{(double-drainage)}$$

$$U_{\mathrm{p}1} = \frac{N_\sigma}{N_\sigma - 1}\left[\frac{1}{h_1}\int_0^{h_1} e^{\omega_1} \mathrm{d}z - \frac{1}{N_\sigma}\right] \quad U_{p2} = \frac{N_\sigma}{N_\sigma - 1}\left[\frac{1}{c h_1}\int_{h_1}^{H} \mathrm{e}^{\omega_2} \mathrm{d}z - \frac{1}{N_\sigma}\right]$$

$$U_p = \frac{N_\sigma}{N_\sigma - 1}\left[\frac{1}{H}\left(\int_0^{h_1} \mathrm{e}^{\omega_1} \mathrm{d}z + \int_{h_1}^{H} \mathrm{e}^{\omega_2} \mathrm{d}z\right) - \frac{1}{N_\sigma}\right]$$

Coefficients and eigen-equation

Single-drainage

$$\omega_1 = -\ln N_\sigma \sum_{m=1}^{\infty} \sin\left(\lambda_m \frac{z}{h_1}\right) \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \frac{2}{\lambda_m (1 + \mu^2 K c A_m^2)}$$

$$\omega_2 = -\ln N_\sigma \sum_{m=1}^{\infty} A_m \cos\left(\mu \lambda_m \frac{H - z}{h_1}\right) \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \frac{2}{\lambda_m (1 + \mu^2 K c A_m^2)}$$

$$A_m = \begin{cases} \dfrac{\sin(\lambda_m)}{\cos(\mu c \lambda_m)} & \cos(\mu c \lambda_m) \neq 0 \\ \dfrac{1}{\mu K} \dfrac{\cos(\lambda_m)}{\sin(\mu c \lambda_m)} & \cos(\mu c \lambda_m) = 0 \end{cases}$$

$$\mu K \sin(\mu c \lambda_m) \sin(\lambda_m) - \cos(\mu c \lambda_m) \cos(\lambda_m) = 0$$

Double-drainage

$$\omega_1 = -\ln N_\sigma \sum_{m=1}^{\infty} \sin\left(\lambda_m \frac{z}{h_1}\right) \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \frac{2(1 + \mu K A_m)}{\lambda_m (1 + \mu^2 K c A_m^2)}$$

$$\omega_2 = -\ln N_\sigma \sum_{m=1}^{\infty} A_m \sin\left(\mu \lambda_m \frac{H - z}{h_1}\right) \mathrm{e}^{-\lambda_m^2 T_{\mathrm{v}1}} \frac{2(1 + \mu K A_m)}{\lambda_m (1 + \mu^2 K c A_m^2)}$$

$$A_m = \begin{cases} \dfrac{\sin(\lambda_m)}{\sin(\mu c \lambda_m)} & \sin(\mu c \lambda_m) \neq 0 \\ -\dfrac{1}{\mu K} \dfrac{\cos(\lambda_m)}{\cos(\mu c \lambda_m)} & \sin(\mu c \lambda_m) = 0 \end{cases}$$

$$\mu K \cos(\mu c \lambda_m) \sin(\lambda_m) + \sin(\mu c \lambda_m) \cos(\lambda_m) = 0$$

$$c = h_2/h_1 \quad \mu = \sqrt{\frac{c_{\mathrm{v}1}}{c_{\mathrm{v}2}}} \quad K = \frac{k_{\mathrm{v}02}}{k_{\mathrm{v}01}} \quad T_{\mathrm{v}1} = \frac{c_{\mathrm{v}1} t}{\mathrm{h}_1} \quad N_\sigma = \frac{\sigma_{\mathrm{f}}^t}{\sigma_0^t} = \frac{\sigma_0^t + q_{\mathrm{u}}}{\sigma_0^t}$$

Table 2
Solutions for the case of linear loading

Excess pore water pressure and average degree of consolidation

$$u_1=\begin{cases}\dfrac{q_\mathrm{u}T}{\mathrm{N}_\sigma-1}(1-\mathrm{e}^{\omega_1}) & t\leqslant t_\mathrm{c}\\ \dfrac{q_\mathrm{u}N_\sigma}{\mathrm{N}_\sigma-1}(1-\mathrm{e}^{\omega_1'}) & t\geqslant t_\mathrm{c}\end{cases}\qquad u_2=\begin{cases}\dfrac{q_\mathrm{u}T}{N_\sigma-1}(1-\mathrm{e}^{\omega_2}) & t\leqslant t_\mathrm{c}\\ \dfrac{q_\mathrm{u}N_\sigma}{N_\sigma-1}(1-\mathrm{e}^{\omega_2'}) & t\geqslant t_\mathrm{c}\end{cases}$$

$$U_s=\begin{cases}\dfrac{1}{\ln N_\sigma}\left\{\ln T-\dfrac{1}{1+\mu^2Kc}\sum_{m=1}^{\infty}\dfrac{2C_1}{\lambda_m^2(1+\mu^2KcA_m^2)}\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\right\} & t\leqslant t_\mathrm{c}\\ \dfrac{1}{\ln N_\sigma}\left\{\ln N_\sigma-\dfrac{1}{1+\mu^2Kc}\sum_{m=1}^{\infty}\dfrac{2C_2}{\lambda_m^2(1+\mu^2KcA_m^2)}\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\right\} & t\geqslant t_\mathrm{c}\quad\text{(single-drainage)}\end{cases}$$

$$U_s=\begin{cases}\dfrac{1}{\ln N_\sigma}\left\{\ln T-\dfrac{1}{1+\mu^2Kc}\sum_{m=1}^{\infty}\dfrac{2(1+\mu KA_m)^2C_1}{\lambda_m^2(1+\mu^2KcA_m^2)}\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\right\} & t\leqslant t_\mathrm{c}\\ \dfrac{1}{\ln N_\sigma}\left\{\ln N_\sigma-\dfrac{1}{1+\mu^2Kc}\sum_{m=1}^{\infty}\dfrac{2(1+\mu KA_m)^2C_2}{\lambda_m^2(1+\mu^2KcA_m)}\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\right\} & t\geqslant t_\mathrm{c}\quad\text{(double-drainage)}\end{cases}$$

$$U_\mathrm{p}=\begin{cases}\dfrac{T}{N_\sigma-1}\left(\dfrac{1}{H}(\int_0^{h_1}\mathrm{e}^{\omega_1}\mathrm{d}z+\int_{h_1}^{H}\mathrm{e}^{\omega_2}\mathrm{d}z)-\dfrac{1}{T}\right) & t\leqslant t_\mathrm{c}\\ \dfrac{\mathrm{N}_\sigma}{\mathrm{N}_\sigma-1}\left(\dfrac{1}{H}(\int_0^{h_1}\mathrm{e}^{\omega_1'}\mathrm{d}z+\int_{h_1}^{H}\mathrm{e}^{\omega_2'}\mathrm{d}z)-\dfrac{1}{N_\sigma}\right) & t\geqslant t_\mathrm{c}\end{cases}$$

Coefficients and eigen-equation
Single-drainage

$$\omega_1=-\sum_{m=1}^{\infty}\sin(\lambda_m\frac{z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2C_1}{\lambda_m(1+\mu^2KcA_m^2)}\qquad \omega_2=-\sum_{m=1}^{\infty}A_m\cos(\mu\lambda_m\frac{H-z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2C_1}{\lambda_m(1+\mu^2KcA_m^2)}$$

$$\omega_1'=-\sum_{m=1}^{\infty}\sin(\lambda_m\frac{z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2C_2}{\lambda_m(1+\mu^2KcA_m^2)}\qquad \omega_2'=-\sum_{m=1}^{\infty}A_m\cos(\mu\lambda_m\frac{H-z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2C_2}{\lambda_m(1+\mu^2KcA_m^2)}$$

$$A_m\begin{cases}\dfrac{\sin(\lambda_m)}{\cos(\mu c\lambda_m)} & \cos(\mu c\lambda_m)\neq 0\\ \dfrac{1}{\mu K}\dfrac{\cos(\lambda_m)}{\sin(\mu c\lambda_m)} & \cos(\mu c\lambda_m)=0\end{cases}$$

$$\mu K\sin(\mu c\lambda_m)\sin(\lambda_m)-\cos(\mu c\lambda_m)\cos(\lambda_m)=0$$

Double-drainage

$$\omega_1=-\sum_{m=1}^{\infty}\sin(\lambda_m\frac{z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2(1+\mu KA_m)C_1}{\lambda_m(1+\mu^2KcA_m^2)}\qquad \omega_2=-\sum_{m=1}^{\infty}A_m\sin(\mu\lambda_m\frac{H-z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2(1+\mu KA_m)C_1}{\lambda_m(1+\mu^2KcA_m^2)}$$

$$\omega_1'=-\sum_{m=1}^{\infty}\sin(\lambda_m\frac{z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2(1+\mu KA_m)C_2}{\lambda_m(1+\mu^2KcA_m^2)}\qquad \omega_2'=-\sum_{m=1}^{\infty}A_m\sin(\mu\lambda_m\frac{H-z}{h_1})\mathrm{e}^{-\lambda_m^2T_{\mathrm{v}1}}\frac{2(1+\mu KA_m)C_2}{\lambda_m(1+\mu^2KcA_m^2)}$$

$$A_m=\begin{cases}\dfrac{\sin(\lambda_m)}{\sin(\mu c\lambda_m)} & \sin(\mu c\lambda_m)\neq 0\\ -\dfrac{1}{\mu K}\dfrac{\cos(\lambda_m)}{\cos(\mu c\lambda_m)} & \sin(\mu c\lambda_m)=0\end{cases}$$

(*continued on next page*)

Table 2 (*continued*)

$\mu K\cos(\mu c\lambda_m)\sin(\lambda_m)+\sin(\mu c\lambda_m)\cos(\lambda_m)=0$

$C_1=\mathrm{e}^{-\frac{\lambda_m^2 T_{\mathrm{vc1}}}{N_\sigma-1}}\left[\ln T+\sum_{k=1}^{\infty}\frac{(\lambda_m^2 T_{\mathrm{vc1}})^k(T^k-1)}{k!k(N_\sigma-1)^k}\right]\quad C_2=\mathrm{e}^{-\frac{\lambda_m^2 T_{\mathrm{vc1}}}{N_\sigma-1}}\left[\ln N_\sigma+\sum_{k=1}^{\infty}\frac{(\lambda_m^2 T_{\mathrm{vc1}})^k(N_\sigma^k-1)}{k!k(N_\sigma-1)^k}\right]$

$c=h_2/h_1\quad \mu=\sqrt{\frac{c_{\mathrm{v1}}}{c_{\mathrm{v1}}}}\quad K=\frac{k_{\mathrm{v02}}}{k_{\mathrm{v02}}}\quad T_{\mathrm{v1}}=\frac{c_{v1}t}{h_1^2}\quad T_{\mathrm{vc1}}=\frac{c_{\mathrm{v1}}t_{\mathrm{c}}}{h_1^2}$

$N_\sigma=\frac{\sigma_f^t}{\sigma_0^t}=\frac{\sigma_0^t+q_u}{\sigma_0^t}\qquad T=\frac{T_{\mathrm{vc1}}+(N_\sigma-1)T_{\mathrm{v1}}}{T_{\mathrm{vc1}}}$

where $B=\sum_{m=1}^{\infty}\frac{2}{M}\sin(\frac{Mz}{H'})\mathrm{e}^{-M^2T_v}$.

Thus when $N_\sigma=1$, the expression of excess pore water pressure in the solution given by Davis and Raymond will change into

$$u=q_{\mathrm{u}}B=q_{\mathrm{u}}\sum_{m=1}^{\infty}\frac{2}{M}\sin(\frac{Mz}{H'})\mathrm{e}^{-M^2T_{\mathrm{v}}} \tag{34}$$

Accordingly, the average degree of consolidation defined in terms of effective stress (i.e. U_{p}) will be given by

$$U_{\mathrm{p}}=1-\frac{1}{H}\int_0^H u\mathrm{d}z=1-\sum_{m=1}^{\infty}\frac{2}{M^2}\mathrm{e}^{-M^2T_{\mathrm{v}}}=U_{\mathrm{s}} \tag{35}$$

Eqs. (34) and (35) are just Terzaghi's solution for linear consolidation.

Since the value of N_σ reflects the magnitude of load [see Eq. (31)], it can be concluded that the discrepancy between linear and nonlinear consolidation is greatly related to the level of load, and the smaller the load level, the smaller the discrepancy.

5. Nonlinear consolidation behavior of double-layered soil

Based on the solution shown above, a computation program has been developed and some diagrams of computed results have been prepared as shown in Figs. 3–8.

In Figs. 3 and 4, curves of the total average degrees of consolidation U_{s} and U_{p} versus time factor $T_{\mathrm{v}}=c_{\mathrm{v1}}t/H^2$ are given for the single-drainage situation, corresponding to $N_\sigma=3.5$, $T_{vc}=c_{\mathrm{v1}}t_c/H^2=0, 0.1, 0.2, 0.5, 1, 2, 5, 10$ respectively.

The curves of U_{s} or U_{p} corresponding to different values of parameters can also be obtained in a similar fashion to Figs. 3 and 4, which can be of practical use for the case of two thin clay layers (for such case, the assumption that the initial effective pressure is constant with depth is approximately applicable).

Like the consolidation characteristics of a one-layer soil [9], the consolidation of a double-layered soil is related to the time factor $T_{\mathrm{v}c}$, the coefficient of consolidation

Table 3
Solution for the case of one-layer soil

Solution for the case of constant loading [2]

$$u = \frac{q_u N_\sigma}{N_\sigma - 1}(1 - N_\sigma^{-B})$$

$$U_s = 1 - \sum_{m=1}^{\infty} \frac{2}{M^2} e^{-M^2 T_v}$$

$$U_p = \frac{N_\sigma}{N_\sigma - 1}\left[\frac{1}{H}\int_0^H N_\sigma^{-B} dz - \frac{1}{N_\sigma}\right]$$

Solution for the case of linear loading [9]

$$u = \begin{cases} \dfrac{q_u T}{N_\sigma - 1}(1 - e^{-B_1}) & t \leqslant t_c \\ \dfrac{q_u N_\sigma}{N_\sigma - 1}(1 - e^{-B_2}) & t \geqslant t_c \end{cases}$$

$$U_s = \begin{cases} \dfrac{1}{\ln N_\sigma}\left(\ln T - \sum_{m=1}^{\infty} \dfrac{2C_1}{M^2} e^{-M^2 T_v}\right) & t \leqslant t_c \\ \dfrac{1}{\ln N_\sigma}\left(\ln N_\sigma - \sum_{m=1}^{\infty} \dfrac{2C_2}{M^2} e^{-M^2 T_v}\right) & t \geqslant t_c \end{cases}$$

$$U_p = \begin{cases} \dfrac{T}{N_\sigma - 1}\left(\dfrac{1}{H}\displaystyle\int_0^H e^{-B_1} dz - \dfrac{1}{T}\right) & t \leqslant t_c \\ \dfrac{N_\sigma}{N_\sigma - 1}\left(\dfrac{1}{H}\displaystyle\int_0^H e^{-B_2} dz - \dfrac{1}{N_\sigma}\right) & t \geqslant t_c \end{cases}$$

Coefficients

$$B = \sum_{m=1}^{\infty}\left[\frac{2}{M}\sin\left(M\frac{z}{H'}\right)e^{-M^2 T_v}\right]$$

$$B_1 = \sum_{m=1}^{\infty}\left[\frac{2C_1}{M}\sin\left(M\frac{z}{H'}\right)e^{-M^2 T_v}\right]$$

$$B_2 = \sum_{m=1}^{\infty}\left[\frac{2C_2}{M}\sin\left(M\frac{z}{H'}\right)e^{-M^2 T_v}\right.$$

$$C_1 = e^{-\frac{M^2 T_{vc}}{N_\sigma - 1}}\left[\ln T + \sum_{k=1}^{\infty} \frac{(M^2 T_{vc})^k (T^k - 1)}{k!k(N_\sigma - 1)^k}\right]$$

$$C_2 = e^{-\frac{M^2 T_{vc}}{N_\sigma - 1}}\left[\ln N_\sigma + \sum_{k=1}^{\infty} \frac{(M^2 T_{vc})^k (N_\sigma^k - 1)}{k!k(N_\sigma - 1)^k}\right]$$

$$M = \frac{2m - 1}{2}\pi \qquad N_\sigma = \frac{q_u + \sigma_0^t}{\sigma_0^t}$$

$$T_v = \frac{c_v t}{H'^2} \qquad T_{vc} = \frac{c_v t_c}{H'^2}$$

$$T = \frac{T_{vc} + (N_\sigma - 1)T_v}{T_{vc}}$$

$H' = H$ (single-drainage)
$H' = H/2$ (double-drainage)

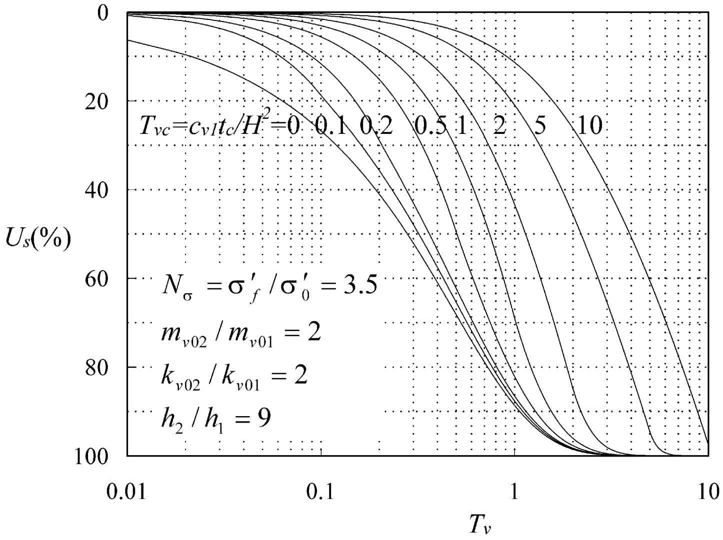

Fig. 3. Variation of U_s vs. T_v curve with T_{vc}.

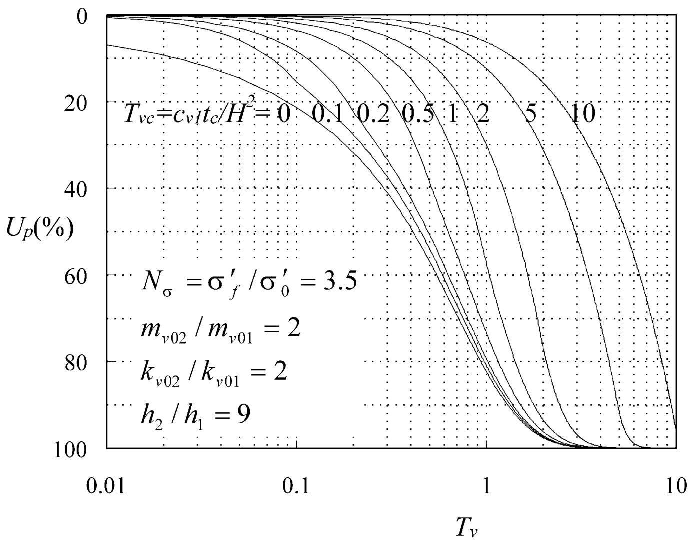

Fig. 4. Variation of U_p vs. T_v curve with T_{vc}.

c_{vi} and the ratio of final effective pressure to initial effective pressure N_σ. Moreover, it varies with parameters μ, K and c, where $K = k_{v02}/k_{v01}$, $c = h_2/h_1$, and $\mu^2 K = m_{v02}/m_{v01}$.

Figs. 5–8 present curves of excess pore water pressure distribution with depth for $T_v = (c_{v1}t)/H^2 = 0.197$,$N_\sigma = 2.5$ ($\sigma'_0 = 80$ kPa, $q_u = 200$ kPa) under constant loading and for the single-drainage situation.

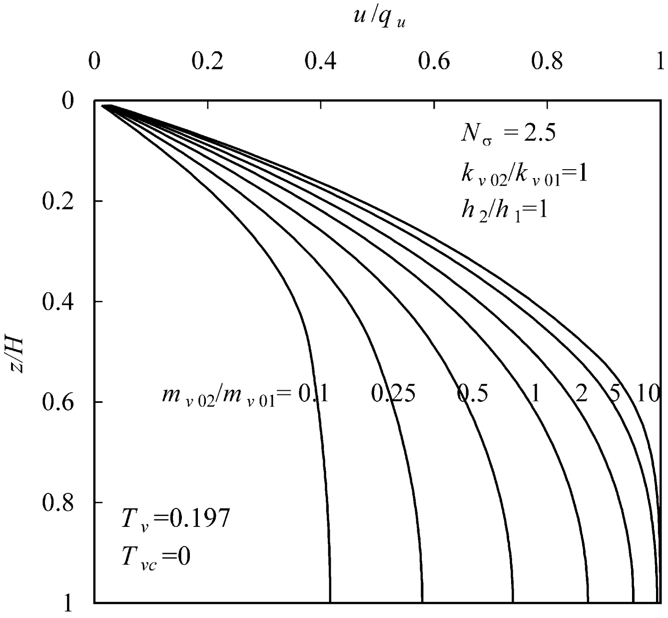

Fig. 5. Influence of the compressibility of the soil on the isochrones of excess pore water pressure.

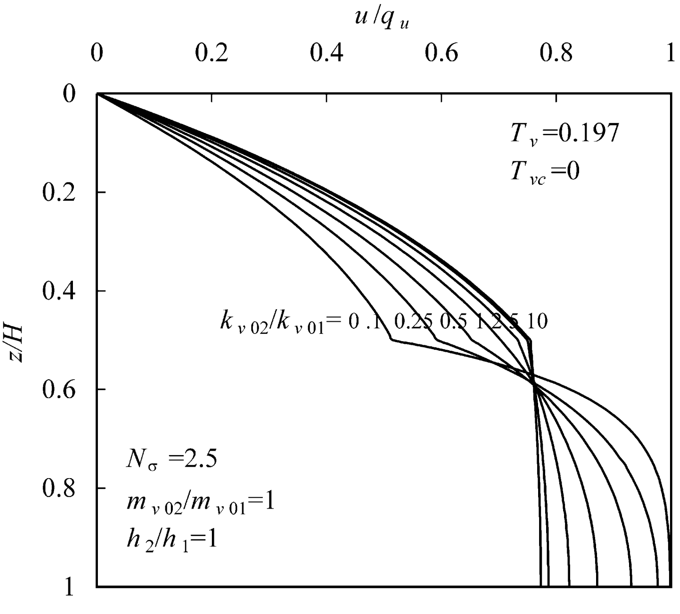

Fig. 6. Influence of the permeability of the soil on the isochrones of excess pore water pressure.

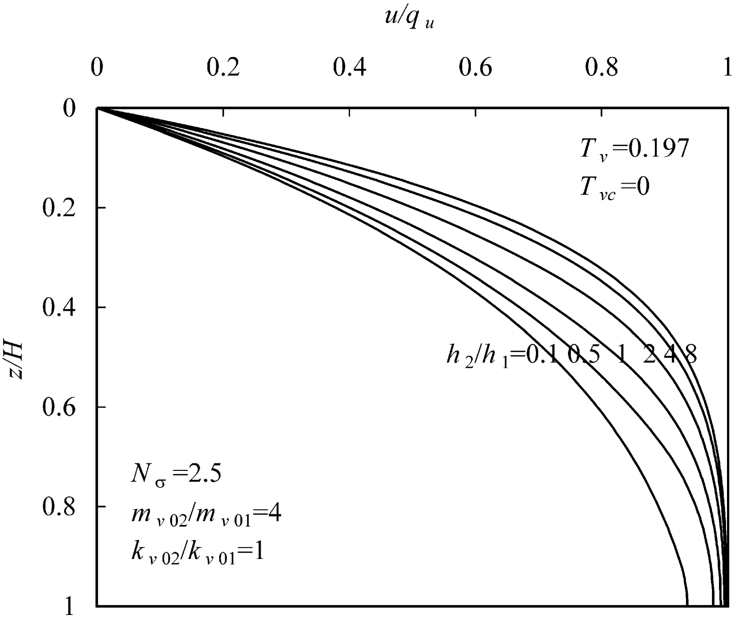

Fig. 7. Variation of the isochrone of excess pore water pressure with the ratio h_2/h_1 ($K=1$, $\mu^2K=4$).

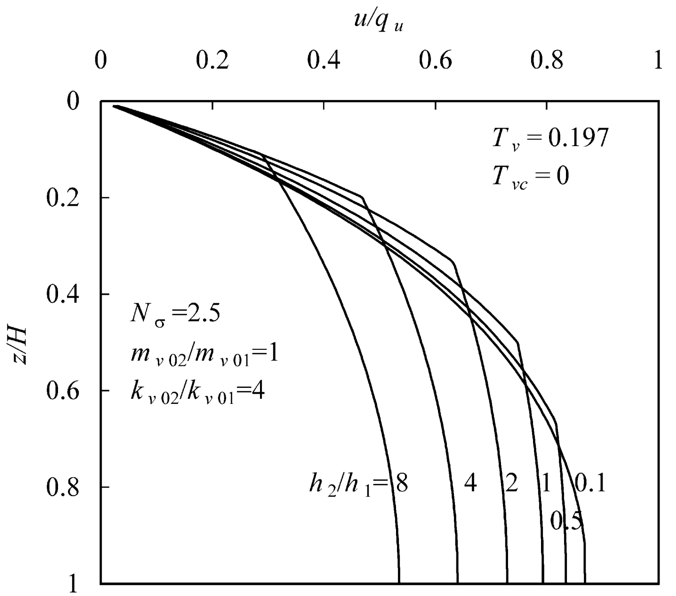

Fig. 8. Variation of the isochrone of excess pore water pressure with the ratio h_2/h_1 ($K=4$, $\mu^2K=1$).

Figs. 5 and 6 show the influences of the compressibility and the permeability of soil on the isochrones of the excess pore water pressure. It can be seen that if the permeability of the two layers is the same (i. e. $K = k_{v02}/k_{v01} = 1$), the greater the ratio of m_{v02}/m_{v01}, the greater the excess pore water pressure, indicating that the increase in the compressibility of soil will delay the rate of consolidation. However, the variation in distribution of the excess pore water pressure with k_{v02}/k_{v01} is more complex. As shown in Fig. 6, the influence of k_{v02}/k_{v01} on the excess pore water pressure in the upper and the lower layers is opposite.

Figs. 7 and 8 show the variation of the excess pore water pressure with different values of c (i.e. h_2/h_1). It can be seen that the thicker the layer with higher compressibility, the slower the rate of dissipation of the excess pore water pressure. But as shown in Fig. 8, it seems that in the layer with higher permeability, the thicker the layer, the smaller the excess pore water pressure, while in the layer of lower permeability, the opposite occurs.

6. Conclusions

1. Based on the assumptions proposed by Davis and Raymond that decrease in permeability is proportional to the decrease in compressibility during consolidation process and the distribution of initial effective pressures is constant with depth, an explicit analytical solution is derived for one-dimensional nonlinear consolidation of double-layered clayey soil. The analytical solutions for nonlinear consolidation so far available are all special cases of this solution.
2. In a nonlinear consolidation, not only is U_s deferent from U_p, but also U_{si} is deferent from U_{pi}.
3. The discrepancy between linear and nonlinear consolidation increases with the increase of load level.
4. Apart from boundary drainage condition, the main factors affecting the rate of nonlinear consolidation of a double-layered soil are the initial coefficients of permeability k_{v0i}, the initial coefficients of volume compressibility m_{v0i} (or the coefficient of consolidation c_{vi}), the thickness of clay layer h_i, the construction time t_c, and the ratio of final effective pressure to initial effective pressure N_σ.

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